Systems Simulation
Chapter 8: Random-Variate Generation

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Introduction

- This chapter deals with procedures for sampling from a variety of widely-used continuous and discrete distributions.
- The purpose of the chapter is to explain and illustrate some widely-used techniques for generating random variates.
- The techniques mentioned here are the inverse-transform technique, the acceptance-rejection technique.
Introduction-cont.

Assumption

- We assume that we have $U[0, 1]$ RVs $R_1, R_2, \ldots$ where

$$f_R(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$F_R(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

Inverse-Transform (IT) Technique

- The IT technique can be used to sample from the exponential, the uniform, the Weibull, the triangular distributions and from empirical distributions.
- It is also the underlying principle for sampling from a wide variety of discrete distributions.
- We will explain it for the exponential distribution and then apply to the others.
- Computationally, it is the most straightforward technique, but not always the most efficient.
IT for the Exponential Distribution

The PDF and CDF of the exponential RV \( X \) are

\[
f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}
\]

\[
F(X) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}
\]

\( \lambda \) can be interpreted as the mean number of occurrences per time unit, and the mean of \( X \) is \( E(X) = \frac{1}{\lambda} \).

The IT can be utilized in principle for any distribution, but it is most useful when the inverse of the CDF \( F(X) \), \( F^{-1} \) is easily computed.

**IT for the Exponential Distribution-cont.**

**Step (1)** Compute the CDF of the RV \( X \).

For the ED, it is \( F(X) = 1 - e^{-\lambda x}, x \geq 0 \).

**Step (2)** Set \( F(X) = R \) on the range of \( X \).

For the ED, it is \( 1 - e^{-\lambda X} = R \) on the range \( x \geq 0 \).

**Step (3)** Solve the equation \( F(X) = R \) for \( X \) in terms of \( R \).

\[
\begin{align*}
1 - e^{-\lambda X} &= R \\
e^{-\lambda X} &= 1 - R \\
-\lambda X &= \ln(1 - R) \\
X &= -\frac{1}{\lambda} \ln(1 - R) \rightarrow X = F^{-1}(R)
\end{align*}
\]
IT for the Exponential Distribution-cont.

Step (4) Generate RNs \( R_1, R_2, \ldots \) and compute the RVs using \( X_i = F^{-1}(R_i) \).

For the ED, it is

\[
X = F^{-1}(R) = -\frac{1}{\lambda} \ln (1 - R) \Rightarrow X_i = -\frac{1}{\lambda} \ln (1 - R_i)
\]

Since both \( R_i \) and \( 1 - R_i \) are uniform, we can write

\[
X_i = -\frac{1}{\lambda} \ln (R_i)
\]

Exponential Distribution-cont.

Example

<table>
<thead>
<tr>
<th>(i)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(R_i)</td>
<td>0.1306</td>
<td>0.0422</td>
<td>0.6597</td>
<td>0.7965</td>
<td>0.7696</td>
</tr>
<tr>
<td>(X_i)</td>
<td>0.1400</td>
<td>0.0431</td>
<td>1.0780</td>
<td>1.5920</td>
<td>1.4680</td>
</tr>
</tbody>
</table>
IT for the Exponential Distribution-cont.

\[ F(x) = 1 - e^{-x} \]

\[ \frac{1}{x} \]

\[ R_2 = \frac{1}{x} \]

\[ R_1 \]

\[ F(x_0) = 0 \]

\[ x_0 = \frac{1}{R_1} n (1 - R_1) \]

Figure: Graphical View of the IT Technique

Building on Exponential Distribution

\[ Y = 2 \sum_{i=1}^{v} X_i \sim C-S (2v) \]

\[ Y = \beta \frac{1}{\alpha} \sum_{i=1}^{\alpha} X_i \sim \text{gamma} (\alpha, \beta) \]

\[ Y = \frac{\sum_{i=1}^{a} X_i}{\sum_{i=1}^{a+b} X_i} \sim \text{beta} (a, b) \]
Systems Simulation  Chapter 8: Random-Variate Generation
Inverse-Transform Technique

**Uniform Distribution**

Step (1) The CDF is given by

\[ f(x) = \begin{cases} 
\frac{1}{b-a}, & a \leq x \leq b \\
0, & \text{otherwise} 
\end{cases} \]

\[ F(X) = \begin{cases} 
0, & x < a \\
\frac{x-a}{b-a}, & a \leq x \leq b \\
1, & x > b 
\end{cases} \]

Step (2) Set \( F(X) = \frac{X-a}{b-a} = R \)

Step (3) Solve for \( X \) in terms of \( R \) to obtain \( X = a + (b-a)R \)

**Weibull Distribution**

Step (1) The CDF is given, when \( \nu = 0 \), by

\[ f(x) = \begin{cases} 
\frac{\beta}{\alpha^{\beta}} x^{\beta-1} e^{-\left(x/\alpha\right)^\beta}, & x \geq 0 \\
0, & \text{otherwise} 
\end{cases} \]

\[ F(X) = 1 - e^{-\left(x/\alpha\right)^\beta}, & x \geq 0 \]

Step (2) Set \( F(X) = 1 - e^{-\left(x/\alpha\right)^\beta} = R \)

Step (3) Solve for \( X \) in terms of \( R \) to obtain

\[ X = \alpha \left[ -\ln(1-R) \right]^{1/\beta} \]
IT Example for the Triangular Distribution

Triangular Distribution (with end points (0, 2) and mode 1)

\[ f(x) = \begin{cases} 
  x, & 0 \leq x \leq 1 \\
  2 - x, & 1 < x \leq 2 \\
  0, & \text{otherwise}
\end{cases} \]

\[ F(X) = \begin{cases} 
  0, & x \leq 0 \\
  \frac{x^2}{2}, & 0 < x \leq 1 \\
  1 - \frac{(2-x)^2}{2}, & 1 < x \leq 2 \\
  1, & x > 2
\end{cases} \]

For 0 \leq X \leq 1,

\[ R = \frac{X^2}{2} \]

and for 1 \leq X \leq 2,

\[ R = 1 - \frac{(2-x)^2}{2} \]

Thus,

\[ X = \begin{cases} 
  \sqrt{2R}, & 0 \leq R \leq \frac{1}{2} \\
  2 - \sqrt{2(1-R)}, & \frac{1}{2} < R \leq 1
\end{cases} \]
Systems Simulation Chapter 8: Random-Variate Generation
Inverse-Transform Technique
Empirical Continuous Distributions

IT Example for Empirical Continuous Distributions

We have the following data: 2.76, 1.83, 1.80, 1.45, 1.24 The data are arranged from smallest to largest. The smallest possible value is assumed to be 0, so we define \( x_0 = 0 \). Each interval has equal probability of \( 1/n = 1/5 \). The slope of the \( i \)th line segment is

\[
a_i = \frac{x(i) - x(i-1)}{i/n - (i-1)/n} = \frac{x(i) - x(i-1)}{1/n}
\]

The inverse CDF, when \( (i-1)/n < R < i/n \), is given by

\[
X = \hat{F}^{-1}(R) = x(i-1) + a_i \left( R - \frac{i-1}{n} \right)
\]

IT Example for Empirical Continuous Distributions-cont.

For example, for \( R_1 = 0.71 \), we have

\[
X_1 = x_4 + a_4 \left( R_1 - \frac{4 - 1}{n} \right) = 1.45 + 1.90(0.71 - 0.60) = 1.66
\]

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_{(i-1)} ) ( &lt; x &lt; x_{(i)} )</th>
<th>Probability</th>
<th>C.P.</th>
<th>Slope</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.00 ( &lt; x \leq 0.80 )</td>
<td>0.2</td>
<td>0.2</td>
<td>4.00</td>
</tr>
<tr>
<td>2</td>
<td>0.80 ( &lt; x \leq 1.24 )</td>
<td>0.2</td>
<td>0.4</td>
<td>2.20</td>
</tr>
<tr>
<td>3</td>
<td>1.24 ( &lt; x \leq 1.45 )</td>
<td>0.2</td>
<td>0.6</td>
<td>1.05</td>
</tr>
<tr>
<td>4</td>
<td>1.45 ( &lt; x \leq 1.83 )</td>
<td>0.2</td>
<td>0.8</td>
<td>1.90</td>
</tr>
<tr>
<td>5</td>
<td>1.83 ( &lt; x \leq 2.76 )</td>
<td>0.2</td>
<td>1.0</td>
<td>4.65</td>
</tr>
</tbody>
</table>
IT Example for Empirical Continuous Distributions-cont.

![CDF for the Example](image)

**Figure**: CDF for the Example

Continuous Distributions without Closed-Form Inverse

- Some distributions do not have a closed form expressions for their CDF or its inverse, such as normal, gamma and beta distributions.
- If we are willing to approximate the inverse CDF, or numerically integrate, we can use the IT method for RV generation.
- A simple approximation, for instance, to the inverse CDF of the normal distribution is proposed by Schmeiser (1979).

\[ X = F^{-1}(R) \approx R^{0.135} - (1 - R)^{0.135} \]

\[ \frac{0.1975}{0.1975} \]
Normal Approximation

\[ \Phi(x) \approx 1 - \phi(x)[b_1 t + b_2 t^2 + b_3 t^4 + b_4 t^4 + b_5 t^6], \quad x > 0 \]

where

\[ t = (1 + px)^{-1} \]

and

\[ p = 0.2316419, \quad b_1 = 0.31938, \quad b_2 = -0.35656, \quad b_3 = 1.78148, \quad b_4 = -1.82125, \quad b_5 = 1.33027 \]

Discrete Distributions

An Empirical Discrete Distribution Example

The PMF and CDF are given as follows:

\[ p(0) = P(X = 0) = 0.50 \]
\[ p(1) = P(X = 1) = 0.30 \]
\[ p(2) = P(X = 2) = 0.20 \]

\[ F(X) = \begin{cases} 
0.0, & x \leq 0 \\
0.5, & 0 \leq x < 1 \\
0.8, & 1 \leq x < 2 \\
1.0, & x \geq 2 
\end{cases} \]
Discrete Distributions

An Empirical Discrete Distribution Example

For generating discrete RVs, the IT technique becomes a table-lookup procedure in this example. For $R = R_1$, if

$$F(x_{i-1}) = r_{i-1} < R \leq r_i = F(X_i)$$

then, set $X_1 = x_i$. We have the following generation scheme here:

$$X = \begin{cases} 
0, & R \leq 0.5 \\
1, & 0.5 < R \leq 0.8 \\
2, & 0.8 < x \leq 1.0 
\end{cases}$$

Figure: CDF for the Example
Discrete Distributions

Discrete Uniform Distribution Example

The PMF and CDF are given as

\[ p(x) = \frac{1}{k}, x = 1, 2, \ldots, k \]

\[ F(x) = \begin{cases} 
0, & x < 1 \\
\frac{1}{k}, & 1 \leq x < 2 \\
\frac{2}{k}, & 2 \leq x < 3 \\
\vdots & \vdots \\
\frac{k-1}{k}, & k-1 \leq x < k \\
1, & k \leq x 
\end{cases} \]

Using \( F(x_{i-1}) = r_{i-1} < R \leq r_i = F(X_i) \), we have the following.

\[ r_{i-1} = \frac{i-1}{k} < R \leq r_i = \frac{i}{k} \]

Solving it for \( i \)

\[ i - 1 < Rk < i \Rightarrow Rk \leq i < Rk + 1 \]

From the above inequality, we obtain

\[ X = \lceil Rk \rceil \]
Acceptance-Rejection Technique

Uniform Distribution

Step (1) Generate a RN $R$.

Step (2) -

(a) If $R \geq 1/4$, accept $X = R$, then, go to Step 3.
(b) If $R < 1/4$, reject $R$, then, go to Step 1.

Step (3) If another RV needed, go to Step 1, otherwise stop.

Acceptance-Rejection Technique

Poisson Distribution

The PMF of a Poisson RV is

$$p(n) = P(N = n) = \frac{e^{-\alpha} \alpha^n}{n!}, n = 0, 1, 2, \ldots$$

We can write

$$N = n \Leftrightarrow A_1 + A_2 + \ldots + A_n \leq 1 < A_1 + A_2 + \ldots + A_n + A_{n+1}$$

Now, we let $A_i = (-1/\alpha) \ln R_i$ (the IT method for the ED)
Acceptance-Rejection Technique

Poisson Distribution

Using the inequality in the previous slide, we obtain

\[
\sum_{i=1}^{n} -\frac{1}{\alpha} \ln(R_i) \leq 1 < \sum_{i=1}^{n+1} -\frac{1}{\alpha} \ln(R_i)
\]

\[
\ln \prod_{i=1}^{n} R_i = \sum_{i=1}^{n} \ln(R_i) \geq -\alpha > \sum_{i=1}^{n+1} \ln(R_i) = \ln \prod_{i=1}^{n+1} R_i
\]

\[
\prod_{i=1}^{n} R_i \geq e^{-\alpha} > \prod_{i=1}^{n+1} R_i
\]

Acceptance-Rejection Technique

Poisson RV Generation Procedure

**Step (1)** Set \( n = 0, P = 1. \)

**Step (2)** Generate a RN \( R_{n+1} \), and replace \( P \) by \( PR_{n+1} \).

**Step (3)** If \( P < e^{-\alpha} \), then, accept \( N = n \); otherwise reject \( n \), increase \( n \) by once, and go to step 2.
Acceptance-Rejection Technique
Non-Stationary Poisson RV Generation Procedure

Step (1) Let \( \lambda^* = \max_{0 \leq t \leq T} \lambda(t) \) be the max of the arrival rate function and set \( t = 0 \) and \( i = 1 \).

Step (2) Generate \( E \) from the exponential distribution with rate \( \lambda^* \) and let \( t = t + E \) (arrival time of the stationary Poisson process).

Step (3) Generate RN \( R \). If \( R \leq \lambda(t)/\lambda^* \), then, let \( \tau_i = t \) and \( i = i + 1 \).

Step (4) Go to step 2.

Acceptance-Rejection Technique
Gamma RV Generation Procedure

Step (1) Compute \( a = 1/(2/\beta - 1)^{1/2} \), \( b = \beta - \ln 4 \).

Step (2) Generate \( R_1 \) and \( R_2 \). Set \( V = R_1/(1 - R_1) \).

Step (3) Compute \( X = \beta V^a \).

Step (4) -
(a) If \( X > b + (\beta a + 1) \ln(V) - \ln(R_2^2 R_1) \), reject \( X \) and return to step 2.
(b) If \( X \leq b + (\beta a + 1) \ln(V) - \ln(R_2^2 R_1) \), accept \( X \).

Step (5) \( X \) has mean and variance both equal to \( \beta \). If it is desired to have mean \( 1/\theta \) and variance \( 1/\beta \theta^2 \), replace \( X \) by \( X/(\beta \theta) \).
Special Properties
Direct Transformation for the Normal and Lognormal Distributions

Consider two standard normal RVs, $Z_1$ and $Z_2$, plotted as a point in the plane as shown in the figure on the next slide, and

\[
Z_1 = B \cos \theta \\
Z_2 = B \sin \theta
\]

It is known that $B^2 = Z_1^2 + Z_2^2$ has a chi-square distribution with 2 degrees of freedom, which is equivalent to an ED with mean 2. So, we can write, $B = \left(-2 \ln R\right)^{1/2}$, and hence,

\[
Z_1 = \left(-2 \ln R_1\right)^{1/2} \cos 2\pi R_2 \\
Z_2 = \left(-2 \ln R_1\right)^{1/2} \sin 2\pi R_2
\]

Figure: Polar Representation of a Pair of Std. Nor. Variables
Special Properties

Convolution Method-Erlang Distribution

An Erlang RV $X$ with parameters $(k, \theta)$ can be shown to be the sum of $k$ independent exponential RVs, $X_i, i = 1, \ldots, k$ each with mean $1/k\theta$. The convolution approach is to generate $X_1, \ldots, X_k$, then, sum them to get $X$. Therefore,

$$X = \sum_{i=1}^{k} -\frac{1}{k\theta} \ln R_i$$

$$= -\frac{1}{k\theta} \ln \left( \prod_{i=1}^{k} R_i \right)$$

More Special Properties-Beta Distribution

Assume that $X_1 \sim G(\beta_1, \theta_1 = 1/\beta_1)$ and $X_2 \sim G(\beta_2, \theta_2 = 1/\beta_2)$, and $X_1$ and $X_2$ are independent. Then,

$$Y = \frac{X_1}{X_1 + X_2}$$

has a beta distribution with $\beta_1$ and $\beta_2$ on the interval $(0, 1)$. If we want $Y$ to be defined on $(a, b)$, then,

$$Y = a + (b - a) \left( \frac{X_1}{X_1 + X_2} \right)$$
Summary

- Reading HW: Chapter 8.
- Chapter 8 Exercises.