Lagrangian Duality & Saddle Point Optimality Conditions

Nonlinear Programming

Fatih Cavdur

to accompany
Nonlinear Programming
by Mokhtar S. Bazara, Hanif D. Sherali and C. M. Shetty

Lagrangian Dual Problem

Given a nonlinear programming problem, there is another nonlinear programming problem closely associated with it. The former is called the primal problem, and the latter is called the Lagrangian dual problem.

Under certain convexity assumptions and suitable constraint qualifications, the primal and dual problems have equal optimal objective values and, hence, it is possible to solve the primal problem indirectly by solving the dual problem.

Lagrangian Dual Problem

Consider the following NLP problem P, which we call the primal problem.

$$\min\{f(\mathbf{x}) : g_i(\mathbf{x}) \leq 0, i = 1, \ldots, m;\ \ h_i(\mathbf{x}) = 0, i = 1, \ldots, l;\ \ \mathbf{x} \in X\}$$

The Lagrangian duality problem D is stated below.

$$\max\{\theta(\mathbf{u}, \mathbf{v}) : \mathbf{u} \geq \mathbf{0}\}$$

where

$$\theta(\mathbf{u}, \mathbf{v}) = \inf\{f(\mathbf{x}) + \sum_{i=1}^{m} u_i g_i(\mathbf{x}) + \sum_{i=1}^{l} v_i h_i(\mathbf{x}) : \mathbf{x} \in X\}$$

Lagrangian Dual Problem

- The problem evaluates θ is sometimes referred to as the Lagrangian dual sub-problem.
- In this problem, inequality and equality constraints have been incorporated in the objective function using the Lagrangian multipliers or dual variables $\mathbf{u}$ and $\mathbf{v}$, respectively.
- Since the dual problem consists of maximizing the infimum, it is sometimes referred to as the max-min dual problem.
- Strictly speaking, we should write D as $\sup\{\theta(\mathbf{u}, \mathbf{v}) : \mathbf{u} \geq \mathbf{0}\}$, rather than $\max\{\theta(\mathbf{u}, \mathbf{v}) : \mathbf{u} \geq \mathbf{0}\}$, since the maximum may not exist.

Lagrangian Dual Problem

The primal and Lagrangian dual problems can be written in the following form using vector notation. The primal problem P:

$$\min\{f(\mathbf{x}) : \mathbf{g}(\mathbf{x}) \leq \mathbf{0}, \mathbf{h}(\mathbf{x}) = \mathbf{0}, \mathbf{x} \in X\}$$

The Lagrangian duality problem D:

$$\max\{\theta(\mathbf{u}, \mathbf{v}) : \mathbf{u} \geq \mathbf{0}\}$$

where

$$\theta(\mathbf{u}, \mathbf{v}) = \inf\{f(\mathbf{x}) + \mathbf{u}^t\mathbf{g}(\mathbf{x}) + \mathbf{v}^t\mathbf{h}(\mathbf{x}) : \mathbf{x} \in X\}$$

Lagrangian Dual Problem

Given an NLP problem, several Lagrangian dual problems can be devised, depending on which constraints are handled as $\mathbf{g}(\mathbf{x}) \leq \mathbf{0}$ and $\mathbf{h}(\mathbf{x}) = \mathbf{0}$ and which constraints are treated by the set X.

This choice can affect both the optimal value of D (as in non-convex situations) and the effort expanded in evaluating and updating the dual function θ during the course of solving the dual problem. Hence, an appropriate selection of the set X must be made, depending on the structure of the problem and the purpose for solving D.

Lagrangian Dual Problem

Example

Consider the following problem with an optimal objective of $f(\mathbf{x}) = 8$ at the point $(x_1, x_2) = (2, 2)$.

$$\min\{f(\mathbf{x}) = x_1^2 + x_2^2 : -x_1 - x_2 + 4 \leq 0, x_1, x_2 \geq 0\}$$

The dual function is given by the following function where the infima are achieved at $x_1 = x_2 = u/2$ if $u \geq 0$ and at $x_1 = x_2 = 0$ if $u < 0$.

$$\begin{aligned}\theta(u) &= \inf\{x_1^2 + x_2^2 + u(-x_1 - x_2 + 4) : x_1, x_2 \geq 0\} \\ &= \inf\{x_1^2 - ux_1 : x_1 \geq 0\} + \inf\{x_2^2 - ux_2 : x_2 \geq 0\} + 4u\end{aligned}$$

Lagrangian Dual Problem

Example

The dual function is given by the following function.

$$\begin{aligned}\theta(u) &= \inf\{x_1^2 + x_2^2 + u(-x_1 - x_2 + 4) : x_1, x_2 \geq 0\} \\ &= \inf\{x_1^2 - ux_1 : x_1 \geq 0\} + \inf\{x_2^2 - ux_2 : x_2 \geq 0\} + 4u\end{aligned}$$

Note that the infima are achieved at $x_1 = x_2 = u/2$ if $u \geq 0$ and at $x_1 = x_2 = 0$ if $u < 0$. Hence,

$$\theta(u) = \begin{cases} -\frac{1}{2}u^2 + 4u, & u \geq 0 \\ 4u, & u < 0 \end{cases}$$

Lagrangian Dual Problem

Example

The dual function is given by the following function.
Note that the infima are achieved at $x_1 = x_2 = u/2$ if $u \geq 0$ and at $x_1 = x_2 = 0$ if $u < 0$. Hence,

$$\theta(u) = \begin{cases} -\frac{1}{2}u^2 + 4u, & u \geq 0 \\ 4u, & u < 0 \end{cases}$$

Note that θ is a convex function, and its maximum over $u \geq 0$ occurs at $\bar{u} = 4$. Note also that the optimal primal and dual objective functions are both equal to 8.

Weak Duality Theorem

Theorem 6.2.1: Weak Duality Theorem

Let $\mathbf{x}$ be a feasible solution to problem P; that is, $\mathbf{x} \in X$, $\mathbf{g}(\mathbf{x}) \leq \mathbf{0}$ and $\mathbf{h}(\mathbf{x}) = \mathbf{0}$. Also, let $(\mathbf{u}, \mathbf{v})$ be a feasible solution to problem D, that is, $\mathbf{u} \geq \mathbf{0}$. Then, $f(x) \geq \theta(\mathbf{u}, \mathbf{v})$.

Weak Duality Theorem

Proof

By the definition of θ, and since $\mathbf{x} \geq \mathbf{0}$, we have

$$\begin{aligned}
\theta(\mathbf{u}, \mathbf{v}) &= \inf\{f(\mathbf{y}) + \mathbf{u}^t\mathbf{g}(\mathbf{y}) + \mathbf{v}^t\mathbf{h}(\mathbf{y}) : \mathbf{y} \in X\} \\
&\leq \inf\{f(\mathbf{x}) + \mathbf{u}^t\mathbf{g}(\mathbf{x}) + \mathbf{v}^t\mathbf{h}(\mathbf{x}) : \mathbf{y} \in X\} \\
&\leq f(\mathbf{x})
\end{aligned}$$

since $\mathbf{u} \geq \mathbf{0}$, $\mathbf{g}(\mathbf{x}) \leq \mathbf{0}$ and $\mathbf{h}(\mathbf{x}) = \mathbf{0}$. This completes the proof.

Strong Duality Theorem

Theorem 6.2.4: Strong Duality Theorem

Let X be a non-empty convex set in R^n, let $f : R^n \to R$ and $g : R^n \to R^m$ be convex, and let $h : R^n \to R^l$ affine. Suppose that the following constraint qualification holds true. There exists $\hat{\mathbf{x}} \in X$ such that $\mathbf{g}(\hat{\mathbf{x}}) \leq \mathbf{0}$ and $\mathbf{h}(\hat{\mathbf{x}}) = \mathbf{0}$, and $\mathbf{0} \in \text{ int } \mathbf{h}(X)$, where $\mathbf{h}(X) = \{\mathbf{h}(\mathbf{x}) : \mathbf{x} \in X\}$. Then,

$$\inf\{f(\mathbf{x}) : \mathbf{x} \in X, \mathbf{g}(\mathbf{x}) \leq \mathbf{0}, \mathbf{h}(\mathbf{x}) = \mathbf{0}\} = \sup\{\theta(\mathbf{u}, \mathbf{v}) : \mathbf{u} \geq \mathbf{0}\}$$

Furthermore, if the infimum is finite, then $\sup\{\theta(\mathbf{u}, \mathbf{v}) : \mathbf{u} \geq \mathbf{0}\}$ is achieved at $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ with $\bar{\mathbf{u}} \geq \mathbf{0}$. If the infimum is achieved at $\bar{\mathbf{x}}$, then $\mathbf{u}^t\mathbf{g}(\mathbf{x}) = 0$.

Strong Duality Theorem

Proof

See the proof on our text on pages 267-268!

Saddle Point Criteria

- Saddle Point of the Lagrangian Function
- Saddle Point Optimality and Absence of a Duality Gap
- Relationships Between the SPC and KKT Conditions
- Saddle Point Optimality Interpretation Using a Perturbation Function

Properties of the Dual Function

- Differentiability of the Dual Function
- Sub-Gradients of the Dual Function
- Ascent and Steepest Ascent Directions

Formulating and Solving the Dual Problem

Cutting Plane or Outer-Linearization Method

Getting the Primal Solution

- Solutions to the Perturbed Primal Problems
- Generating Primal Feasible Solutions in the Convex Case

Linear and Quadratic Programs

- Linear Programs
- Quadratic Programs

Summary

- Lagrangian Dual Problem
- Duality Theorems and Saddle Point Optimality Conditions
- Properties of the Dual Function
- Formulating and Solving the Dual Problem
- Getting the Primal Solution
- Linear and Quadratic Programs

Thanks! Questions?