Constraint Qualifications

Nonlinear Programming

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to accompany
Nonlinear Programming
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Cone of Tangents

In Chapter 4 (Section 4.2), we showed that local optimality implies that $F \cap G_0 = \varnothing$, which in turn implies the Fritz John conditions.

Under the linear independence constraint qualification, or, more generally, under the constraint qualification $G_0 \neq \varnothing$, we deduced that the Fritz John conditions can only be satisfied if the Lagrangian multiplier associated with the objective function is positive. This led to the KKT conditions. This process is summarized in the following flowchart.

$$\begin{array}{c} \text{Local} \\ \text{Optitmality} \end{array} \underset{\text{T-4.2.2}}{\rightarrow} F_0 \cap D = \varnothing \underset{\text{T-4.2.8}}{\rightarrow} \begin{array}{c} \text{FJ} \\ \text{Condidions} \end{array} \underset{\text{CQ}}{\rightarrow} \begin{array}{c} \text{KKT} \\ \text{Conditions} \end{array}$$

Cone of Tangents

In this section, we derive the KKT conditions directly without first obtaining the FJ conditions. Using the CQ $T = G'$, we get $F_0 \cap G' = \varnothing$. Using Farkas's theorem, this statement gives the KKT conditions as follows:

$$\begin{array}{c} \text{Local} \\ \text{Optitmality} \end{array} \underset{\text{T-5.1.2}}{\rightarrow} F_0 \cap T = \varnothing \underset{T = G'}{\rightarrow} F_0 \cap G' = \varnothing \underset{\text{FT}}{\rightarrow} \begin{array}{c} \text{KKT} \\ \text{Conditions} \end{array}$$

Definition 5.1.1

Definition

Let S be a non-empty set in R^n, and let $\bar{\mathbf{x}} \in \text{cl}S$. The cone of tangents of S at $\bar{\mathbf{x}}$, denoted by T, is the set of all directions $\mathbf{d}$ such that $\mathbf{d} = \lim_{k\to\infty} \lambda_k(\mathbf{x}_k - \bar{\mathbf{x}})$ where $\lambda_k > 0$, $\mathbf{x}_k \in S$ for each k, and $\mathbf{x}_k \to \mathbf{x}$.

From the above definition it is clear that $\mathbf{d}$ belongs to the cone of tangents if there is a feasible sequence $\{\mathbf{x}_k\}$ converging to $\bar{\mathbf{x}}$ such that the directions $\mathbf{x}_k - \bar{\mathbf{x}}$ converge to $\mathbf{d}$.

Theorem 5.1.2

The following theorem shows that for the problem to minimize $f(\mathbf{x})$ subject to $\mathbf{x} \in S$, $F_0 \cap T = \varnothing$ is a necessary condition for optimality.

Theorem

Let S be a non-empty set in R^n, and let $\bar{\mathbf{x}} \in S$. Furthermore, suppose that $f : R^n \to R$ is differentiable at $\bar{\mathbf{x}}$. If $\bar{\mathbf{x}}$ locally solves the problem to minimize $f(\mathbf{x})$ subject to $\mathbf{x} \in S$, $F_0 \cap T = \varnothing$, where $F_0 = \{\mathbf{d} : \nabla f(\bar{\mathbf{x}})^t\mathbf{d} < 0\}$.

Theorem 5.1.3: Abadie Constraint Qualification

Theorem: KKT Necessary Conditions

Let S be a non-empty set in R^n, and let $f : R^n \rightarrow R$ and $g_i : R^n \rightarrow R$ for $i = 1, \ldots, m$. Consider the problem to minimize $f(\mathbf{x})$ subject to $\mathbf{x} \in X$ and $g_i(\mathbf{x}) \leq 0$ for $i = 1, \ldots, m$. Let $\bar{\mathbf{x}}$ be a feasible solution, and let $I = \{i : g_i(\bar{\mathbf{x}}) = 0\}$. Suppose that f and g_i for $i \in I$ are differentiable at $\bar{\mathbf{x}}$. Furthermore, suppose that the QC $T = G'$ holds true where T is cone of tangents of the feasible region and $G' = \{\mathbf{d} : \nabla g_i(\bar{\mathbf{x}})^t \leq 0, i \in I\}$. If $\bar{\mathbf{x}}$ is a local optimal solution, there exists non-negative scalars u_i for $i \in I$ such that

$$\nabla f(\bar{\mathbf{x}}) + \sum_{i \in I} u_i \nabla g_i(\bar{\mathbf{x}}) = 0$$

Summary

- Cone of Tangents
- Other Constraint Qualifications
- Problems with Inequality and Equality Constraints

Thanks! Questions?