FJ and KKT Optimality Conditions

Nonlinear Programming

Fatih Cavdur

to accompany
Nonlinear Programming
by Mokhtar S. Bazara, Hanif D. Sherali and C. M. Shetty

Local and Global Optimum

Definition

Consider the problem of minimizing $f(\mathbf{x})$ over R^n, and let $\bar{\mathbf{x}} \in R^n$.
If $f(\mathbf{x}) \geq f(\bar{\mathbf{x}})$ for all $\mathbf{x} \in R^n$, $\bar{\mathbf{x}}$ is called a global minimum.
If there exists an ϵ-neighborhood $N_\epsilon(\bar{\mathbf{x}})$ around $\bar{\mathbf{x}}$ such that $f(\mathbf{x}) \geq f(\bar{\mathbf{x}})$ for each $\mathbf{x} \in N_\epsilon(\bar{\mathbf{x}})$, $\bar{\mathbf{x}}$ is called a local minimum, while if $f(\mathbf{x}) < f(\bar{\mathbf{x}})$ for all $\mathbf{x} \in N_\epsilon(\bar{\mathbf{x}})$, $\mathbf{x} \neq \bar{\mathbf{x}}$, for some $\epsilon > 0$, $\bar{\mathbf{x}}$ is called a strict local minimum.
Clearly, a global minimum is also a local minimum.

Necessary Optimality Conditions

Theorem

Suppose that $f : R^n \to R$ is differentiable at $\bar{\mathbf{x}}$. If there is a vector $\mathbf{d}$ such that $\nabla f(\bar{\mathbf{d}})^t \mathbf{d} < 0$, there exists a $\delta > 0$ such that $f(\bar{\mathbf{x}} + \lambda \mathbf{d}) < f(\bar{\mathbf{x}})$ for each $\delta \in (0, \delta)$, so that $\mathbf{d}$ is a descent direction of f at $\bar{\mathbf{x}}$.

Corollary

Suppose that $f : R^n \to R$ is differentiable at $\bar{\mathbf{x}}$. If $\bar{\mathbf{x}}$ is a local minimum, $\nabla f(\bar{\mathbf{x}}) = \mathbf{0}$.

Necessary Optimality Conditions

Theorem

Suppose that $f : R^n \to R$ is twice differentiable at $\bar{\mathbf{x}}$. If $\bar{\mathbf{x}}$ is a local minimum, $\nabla f(\bar{\mathbf{x}}) = \mathbf{0}$ and $\mathbf{H}(\bar{\mathbf{x}})$ is positive semidefinite.

Sufficient Optimality Conditions

Theorem

Suppose that $f : R^n \to R$ is twice differentiable at $\bar{\mathbf{x}}$. If $\nabla f(\bar{\mathbf{x}}) = \mathbf{0}$ and $\mathbf{H}(\bar{\mathbf{x}})$ is positive, $\bar{\mathbf{x}}$ is a strict local minimum.

Sufficient Optimality Conditions

Theorem

Let $f : R^n \to R$ be pseudoconvex at $\bar{\mathbf{x}}$. Then $\bar{\mathbf{x}}$ is a global minimum if and only if $\nabla f(\bar{\mathbf{x}}) = \mathbf{0}$.

Sufficient Optimality Conditions

Theorem

Let $f : R^n \to R$ be infinitely differentiable univariate function. Then $\bar{x} \in R$ is a local minimum if and only if either $f^{(j)}(\bar{x}) = 0$ for all $j = 1, 2, \ldots$ or else there exists a even $n \geq 2$ such that $f^{(n)}(\bar{x}) > 0$ while $f^{(j)}(\bar{x}) = 0$ for all $1 \leq j < n$.

Optimality Conditions

Theorem

Consider the problem to minimize $f(\mathbf{x})$ subject to $\mathbf{x} \in S$ where $f : R^n \to R$ and S is a nonempty set in R^n. Suppose that f is differentiable at $\mathbf{x} \in S$. If $\bar{\mathbf{x}}$ is a local optimal solution, $F_0 \cap D = \emptyset$, where $F_0 = \{\mathbf{d} : \nabla f(\bar{\mathbf{x}})^t \mathbf{d} < 0\}$ and D is the cone of feasible directions of S at $\bar{\mathbf{x}}$. Conversely, suppose that $F_0 \cap D = \emptyset$, f is pseudoconvex at $\bar{\mathbf{x}}$, and that there exists an ϵ-neighborhood $N_\epsilon(\bar{\mathbf{x}})$, $\epsilon > 0$, such that $\mathbf{d} = (\mathbf{x} - \bar{\mathbf{x}}) \in D$ for any $\mathbf{x} \in S \cap N_\epsilon(\bar{\mathbf{x}})$. Then $\bar{\mathbf{x}}$ is a local minimum of f.

Optimality Conditions

Theorem

Consider problem P to minimize $f(\mathbf{x})$ subject to $\mathbf{x} \in X$ and $g_i(\mathbf{x}) \leq 0$ for $i = 1, \ldots, m$, where X is a nonempty set in R^n, $f : R^n \to R$ and $g_i : R^n \to R$ for $i = 1, \ldots, m$. Let $\bar{\mathbf{x}}$ be a feasible point, and denote $I = \{i : g_i(\bar{\mathbf{x}}) = 0\}$. Furthermore, suppose that f and g_i for $i \in I$ are differentiable at $\bar{\mathbf{x}}$ and that g_i for $i \notin I$ are continuous at $\bar{\mathbf{x}}$. If $\bar{\mathbf{x}}$ is a local optimal solution, $F_0 \cap G_0 = \emptyset$, where $F_0 = \{\mathbf{d} : \nabla f(\bar{\mathbf{x}})^t \mathbf{d} < 0\}$ and $G_0 = \{\mathbf{d} : \nabla g_i(\bar{\mathbf{x}})^t \mathbf{d} < 0, \forall i \in I\}$. Conversely, if $F_0 \cap G_0 = \emptyset$, and if f is pseudoconvex at $\bar{\mathbf{x}}$ and g_i, $i \in I$, are strictly pseudoconvex over some ϵ-neighborhood of $\bar{\mathbf{x}}$, $\bar{\mathbf{x}}$ is a local minimum.

Fritz-John Necessary Conditions

Theorem: Fritz-John Necessary Conditions

Consider problem P to minimize $f(\mathbf{x})$ subject to $\mathbf{x} \in X$ and $g_i(\mathbf{x}) \leq 0$ for $i = 1, \ldots, m$, where X is a nonempty set in R^n, $f : R^n \to R$ and $g_i : R^n \to R$ for $i = 1, \ldots, m$. Let $\bar{\mathbf{x}}$ be a feasible point, and denote $I = \{i : g_i(\bar{\mathbf{x}}) = 0\}$. Furthermore, suppose that f and g_i for $i \in I$ are differentiable at $\bar{\mathbf{x}}$ and that g_i for $i \notin I$ are continuous at $\bar{\mathbf{x}}$.

Fritz-John Necessary Conditions

Theorem: Fritz-John Necessary Conditions (cont.)

If $\bar{\mathbf{x}}$ is a local optimal solution, there exists scalars u_0 and u_i, for $i \in I$ such that

$$u_0 \nabla f(\bar{\mathbf{x}}) + \sum_{i \in I} u_i \nabla g_i(\bar{\mathbf{x}}) = \mathbf{0}$$

$$u_0, u_i \geq 0, \quad \forall i \in I$$

$$(u_0, \mathbf{u}_I) \neq (0, \mathbf{0})$$

Fritz-John Necessary Conditions

Theorem: Fritz-John Necessary Conditions (cont.)

Furthermore, g_i for $i \notin I$ are differentiable at $\bar{\mathbf{x}}$, the foregoing conditions can be written in the following equivalent form:

$$\begin{aligned} u_0 \nabla f(\bar{\mathbf{x}}) + \sum_{i \in I} u_i \nabla g_i(\bar{\mathbf{x}}) &= \mathbf{0} \\ u_i g_i(\bar{\mathbf{x}}) &= 0, \quad \forall i \in I \\ u_0, u_i &\geq 0, \quad \forall i \in I \\ (u_0, \mathbf{u}_I) &\neq (0, \mathbf{0}) \end{aligned}$$

Fritz-John Sufficient Conditions

Theorem: Fritz-John Sufficient Conditions

Consider problem P to minimize $f(\mathbf{x})$ subject to $\mathbf{x} \in X$ and $g_i(\mathbf{x}) \leq 0$ for $i = 1, \ldots, m$, where X is a nonempty set in R^n, $f : R^n \to R$ and $g_i : R^n \to R$ for $i = 1, \ldots, m$. Let $\bar{\mathbf{x}}$ be an FJ solution, and denote $I = \{i : g_i(\bar{\mathbf{x}}) = 0\}$. Define S as the relaxed feasible region for problem P in which the nonbinding constraints are dropped.

Fritz-John Sufficient Conditions

Theorem: Fritz John Sufficient Conditions (cont.)

If there exists an ϵ-neighborhood $N_\epsilon(\bar{\mathbf{x}})$, $\epsilon > 0$, such that f is pseudoconvex over $N_\epsilon(\bar{\mathbf{x}}) \cap S$, $\bar{\mathbf{x}}$ is a local minimum for Problem P.

Fritz-John Sufficient Conditions

Theorem: Fritz John Sufficient Conditions (cont.)

If f is pseudoconvex at $\bar{\mathbf{x}}$, and if g_i, $i \in I$ are both strictly pseudoconvex and quasiconvex at $\bar{\mathbf{x}}$, $\bar{\mathbf{x}}$ is a global optimal solution for Problem P. In particular, if these generalized convexity assumptions hold true only by restricting the domain of f to $N_\epsilon(\bar{\mathbf{x}})$ for some $\epsilon > 0$, $\bar{\mathbf{x}}$ is a local minimum for Problem P.

KKT Necessary Conditions

Theorem: KKT Necessary Conditions

Consider problem P to minimize $f(\mathbf{x})$ subject to $\mathbf{x} \in X$ and $g_i(\mathbf{x}) \leq 0$ for $i = 1, \ldots, m$, where X is a nonempty set in R^n, $f : R^n \to R$ and $g_i : R^n \to R$ for $i = 1, \ldots, m$. Let $\bar{\mathbf{x}}$ be a feasible point, and denote $I = \{i : g_i(\bar{\mathbf{x}}) = 0\}$. Furthermore, suppose that f and g_i for $i \in I$ are differentiable at $\bar{\mathbf{x}}$ and that g_i for $i \notin I$ are continuous at $\bar{\mathbf{x}}$. Furthermore, suppose that $\nabla g_i(\bar{\mathbf{x}})$ for $i \in I$ are linearly independent.

KKT Necessary Conditions

Theorem: KKT Necessary Conditions (cont.)

If $\bar{\mathbf{x}}$ is a local optimal solution, there exists scalars u_i, for $i \in I$ such that

$$\nabla f(\bar{\mathbf{x}}) + \sum_{i \in I} u_i \nabla g_i(\bar{\mathbf{x}}) = \mathbf{0}$$

$$u_i \geq 0, \quad \forall i \in I$$

KKT Necessary Conditions

Theorem: KKT Necessary Conditions (cont.)

Furthermore, g_i for $i \notin I$ are differentiable at $\bar{\mathbf{x}}$, the foregoing conditions can be written in the following equivalent form:

$$\nabla f(\bar{\mathbf{x}}) + \sum_{i \in I} u_i \nabla g_i(\bar{\mathbf{x}}) = \mathbf{0}$$

$$u_i g_i(\bar{\mathbf{x}}) = 0, \quad \forall i \in I$$

$$u_i \geq 0, \quad \forall i \in I$$

KKT Sufficient Conditions

Theorem: KKT Sufficient Conditions

Consider problem P to minimize $f(\mathbf{x})$ subject to $\mathbf{x} \in X$ and $g_i(\mathbf{x}) \leq 0$ for $i = 1, \ldots, m$, where X is a nonempty set in R^n, $f : R^n \to R$ and $g_i : R^n \to R$ for $i = 1, \ldots, m$. Let $\bar{\mathbf{x}}$ be a KKT solution, and denote $I = \{i : g_i(\bar{\mathbf{x}}) = 0\}$. Define S as the relaxed feasible region for problem P in which the nonbinding constraints are dropped.

KKT Sufficient Conditions

Theorem: KKT Sufficient Conditions (cont.)

If there exists an ϵ-neighborhood $N_\epsilon(\bar{\mathbf{x}})$, $\epsilon > 0$ such that f is pseudoconvex over $N_\epsilon(\bar{\mathbf{x}}) \cap S$ and g_i, $i \in I$, are differentiable at $\bar{\mathbf{x}}$ and are quasiconvex over $N_\epsilon(\bar{\mathbf{x}}) \cap S$, $\bar{\mathbf{x}}$ is a local minimum for Problem P.

KKT Sufficient Conditions

Theorem: KKT Sufficient Conditions (cont.)

If f is pseudoconvex at $\bar{\mathbf{x}}$, and if g_i, $i \in I$ are differentiable and quasiconvex at $\bar{\mathbf{x}}$, $\bar{\mathbf{x}}$ is a global optimal solution for Problem P. In particular, if this assumption holds true with the domain of the feasible restriction to $N_\epsilon(\bar{\mathbf{x}})$ for some $\epsilon > 0$, $\bar{\mathbf{x}}$ is a local minimum for Problem P.

Problems with Equality Constraints

- Problems with Inequality Constraints
- FC Necessary Conditions
- FC Sufficient Conditions
- KKT Necessary Conditions
- KKT Sufficient Conditions

Second-Order NOC and SOC for Constrained Problems

- Second Order Conditions
- KKT Second-Order Sufficient Conditions
- KKT Second-Oder Necessary Conditions

Summary

- Unconstrained Problems
- Problems with Inequality Constraints
- Problems with Inequality and Equality Constraints
- Second-Order Necessary and Sufficient Optimality Conditions for Constrained Problems

Thanks! Questions?