Convex Analysis - Convex Sets

Nonlinear Programming

Fatih Cavdur

to accompany
Nonlinear Programming
by Mokhtar S. Bazara, Hanif D. Sherali and C. M. Shetty

Convex Hulls

Closure and Interior of a Set

Weierstrass's Theorem

Separation and Support of Sets

Convex Cones and Polarity

Polyhedral Sets, Extreme Points and Extreme Directions

Linear Programming and the Simplex Method

Summary

Convex Hulls

In this section, we first introduce the notions of convex sets and convex hulls. We then demonstrate that any point in the convex hull of a set S can be represented in terms of $n+1$ points in the set S.

Convex Hulls

Convex Set

A set S is said to be convex if the line segment joining any two points of the set also belong to the set. In other words, if $\mathbf{x}_1$ and $\mathbf{x}_2$ are in S, then $\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$, must also belong to S for each $\lambda \in [0,1]$.

Convex Combination

$\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$, where $\lambda \in [0,1]$, are referred to as convex combinations of $\mathbf{x}_1$ and $\mathbf{x}_2$. Inductively, weighted averages of the form $\sum_j \lambda_j \mathbf{x}_j$, where $\sum_j \lambda_j = 1$, $\lambda_j \geq 0$, $j = 1, \ldots, k$, are also called convex combinations of $\mathbf{x}_1, \ldots, \mathbf{x}_k$.

Convex Hulls

Affine Combination

A combination where the non-negativity conditions of λ_j ($\lambda_j \geq 0$), $j = 1, \ldots, k$ is dropped is known as an affine combination. That is, $\sum_j \lambda_j \mathbf{x}_j$, where $\sum_j \lambda_j = 1$, $j = 1, \ldots, k$, are called affine combinations of $\mathbf{x}_1, \ldots, \mathbf{x}_k$.

Linear Combination

A combination where $\lambda_j \in \mathbb{R}$, $j = 1, \ldots, k$ is known as a linear combination. That is, $\sum_j \lambda_j \mathbf{x}_j$, where $\lambda_j \in \mathbb{R}$, $j = 1, \ldots, k$, are called linear combinations of $\mathbf{x}_1, \ldots, \mathbf{x}_k$.

Convex Hulls

Lemma

Let S_1 and S_2 be convex sets in $\mathbb{R}^n$. Then,

- $S_1 \cap S_2$ is convex.
- $S_1 \oplus S_2 = \{\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_1 \in S_1, \mathbf{x}_2 \in S_2\}$ is convex.
- $S_1 \ominus S_2 = \{\mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 \in S_1, \mathbf{x}_2 \in S_2\}$ is convex.

Convex Hulls

Definition: Convex Hulls

Let S be an arbitrary set in $\mathbb{R}^n$. The convex hull of S, denoted by $\text{conv}(S)$, is the collection of all all convex combinations of S. In other words, $\mathbf{x} \in \text{conv}(S)$ if and only if $\mathbf{x}$ can be represented as

$$\mathbf{x} = \sum_{j=1}^{k} \lambda_j \mathbf{x}_j$$

$$\sum_{j=1}^{k} \lambda_j = 1$$

where $k \in \mathbb{Z}^+$, $\lambda_j \geq 0, \forall j$ and $\mathbf{x}_j \in S, \forall j$.

Convex Hulls

We note that $\text{conv}(S)$ is the minimal (tightest enveloping) convex set that contains S.

Lemma

Let S be an arbitrary set in $\mathbb{R}^n$. Then, $\text{conv}(S)$ is the smallest convex set containing S. Indeed, $\text{conv}(S)$ is the intersection of all convex sets containing S.

Similarly, we can define the affine hull of S as the collection of all affine combinations of points in S. This is the smallest dimensional affine subspace that contains S. Similarly, the linear hull of S is the collection of all linear combinations of points in S.

Polytope and Simplex

We can now define a polytope and simplex.

Definition: Polytope and Simplex

The convex hull of a finite number of points $\mathbf{x}_1, \ldots, \mathbf{x}_{k+1}$ is called a polytope. If $\mathbf{x}_1, \ldots, \mathbf{x}_{k+1}$ are affinely independent, which means that $\mathbf{x}_2 - \mathbf{x}_1, \ldots, \mathbf{x}_{k+1} - \mathbf{x}_1$ are linearly independent, then $\text{conv}(\mathbf{x}_1, \ldots, \mathbf{x}_{k+1})$, the convex hull of $\mathbf{x}_1, \ldots, \mathbf{x}_{k+1}$, is called a simplex having vertices $\mathbf{x}_1, \ldots, \mathbf{x}_{k+1}$.

Caratheodory Theorem

By definition, a point in the convex hull of a set can be represented as a convex combination of a finite number of points in the set. The following theorem shows that any point $\mathbf{x}$ in the convex hull of a set S can be represented as a convex combination of, at most, $n+1$ points in S. The theorem is trivially true for $\mathbf{x} \in S$.

Caratheodory Theorem

Theorem: Caratheodory Theorem

Let S be an arbitrary set in $\mathbb{R}^n$. If $\mathbf{x} \in \text{conv}(S)$, then $\mathbf{x} \in \text{conv}(\mathbf{x}_1, \ldots, \mathbf{x}_{n+1})$, where $\mathbf{x} \in S$ for $j = 1, \ldots, n+1$. In other words, x can be represented as

$$\mathbf{x} = \sum_{j=1}^{n+1} \lambda_j \mathbf{x}_j$$

$$\sum_{j=1}^{n+1} \lambda_j = 1$$

where $\lambda_j \geq 0, \forall j$ and $\mathbf{x}_j \in S, \forall j$.

Caratheodory Theorem

Proof: Caratheodory Theorem

Give the proof here!

Closure and Interior of a Set

In this section we develop some topological properties of sets in general and of convex sets in particular. As a preliminary, given a point $\mathbf{x}$ in $\mathbb{R}^n$, an ϵ-neighborhood around it is the set $N_\epsilon(\mathbf{x}) = \{\mathbf{y} : ||\mathbf{y} - \mathbf{x}|| < \epsilon\}$. Let us first review the definitions of closure, interior, and boundary of an arbitrary set in $\mathbb{R}^n$, using the concept of an ϵ-neighborhood.

Closure and Interior of a Set

Definition: Closure

Let S be an arbitrary set in $\mathbb{R}^n$. A point $\mathbf{x}$ is said to be in the closure of S, denoted by $\text{cl}(S)$, if $S \cap N_\epsilon(\mathbf{x}) \neq \emptyset$ for every $\epsilon > 0$.

Definition: Closed Set

If $S = \text{cl}(S)$, S is called closed.

Definition: Interior

A point $\mathbf{x}$ is said to be in the interior of S, denoted by $\text{int}(S)$, if $N_\epsilon(\mathbf{x}) \subset S$ for some $\epsilon > 0$.

Closure and Interior of a Set

Definition: Solid Set

A solid set $S \subseteq \mathbb{R}^n$ is one having a nonempty interior.

Definition: Open Set

If $S = \text{int}(S)$, S is called open.

Definition: Boundary

$\mathbf{x}$ is said to be in the boundary of S, denoted by ∂S, if $N_\epsilon(\mathbf{x})$ contains at least one point in S and one point not in S for every $\epsilon > 0$.

Closure and Interior of a Set

Definition: Bounded Set

A set S is bounded if it can be contained in a ball of a sufficiently large radius.

Definition: Compact Set

A compact set is one that is both closed and bounded.

Note that the complement of an open set is a closed set and vice versa, and the boundary points of any set and its complement are the same.

Closure and Interior of a Set

- A set S is closed iif it contains all its boundary points.
- The smallest closed set containing S is $\mathrm{cl}(S) = S \cup \partial S$.
- A set is open iif it does not contain any of its boundary points, that is, $\partial S \cap S = \emptyset$.
- Clearly, a set may be neither open nor closed, and the only sets in $\mathbb{R}^n$ that are both open and closed are the empty set and $\mathbb{R}^n$ itself.

Closure and Interior of a Set

- Note that any point $\mathbf{x} \in S$ must be either an interior or a boundary point of S.
- However, $S \neq \mathrm{int}(S) \cup \partial S$, since S need not contain its boundary points.
- We however have $\mathrm{int}(S) = S - \partial S$ since $\mathrm{int}(S) \subseteq S$, while $\partial S \neq S - \mathrm{int}(S)$ necessarily.

Closure and Interior of a Set

- Note that any point $\mathbf{x} \in S$ must be either an interior or a boundary point of S.
- However, $S \neq \text{int}(S) \cup \partial S$, since S need not contain its boundary points.
- We however have $\text{int}(S) = S - \partial S$ since $\text{int}(S) \subseteq S$, while $\partial S \neq S - \text{int}(S)$ necessarily.

Closure and Interior of a Set

Theorem

Let S be a convex set in $\mathbb{R}^n$ with a nonempty interior. Let $\mathbf{x}_1 \in \text{cl}(S)$ and $\mathbf{x}_2 \in \text{int}(S)$. Then $\lambda \mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 \in \text{int}(S)$ for each $\lambda \in (0, 1)$.

Closure and Interior of a Set

Proof

Here is the proof.

Closure and Interior of a Set

Corollary 1

Let S be a convex set. Then $\text{int}(S)$ is convex.

Corollary 2

Let S be a convex set with a nonempty interior. Then $\text{cl}(S)$ is convex.

Closure and Interior of a Set

Corollary 3

Let S be a convex set with a nonempty interior. Then $\text{cl}(\text{int}(S)) = \text{cl}(S)$.

Corollary 4

Let S be a convex set with a nonempty interior. Then $\text{int}(\text{cl}(S)) = \text{int}(S)$.

Weierstrass's Theorem

This result relates to the existence of minimizing solution for an optimization problem.

- We say that $\bar{\mathbf{x}}$ is a minimizing solution for the problem $\min\{f(\mathbf{x}), \mathbf{x} \in S\}$, provided that $\bar{\mathbf{x}} \in S$ and $f(\bar{\mathbf{x}}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in S$. In such a case we say that a minimum exists.
- On the other hand, we say that $\alpha = \inf\{f(\mathbf{x}), \mathbf{x} \in S\}$, if α is the greatest lower bound of f on S; that is, $\alpha \leq f(\mathbf{x})$ for all $\mathbf{x} \in S$ and there is no $\bar{\alpha} > \alpha$ such that $\bar{\alpha} \leq f(\mathbf{x})$ for all $\mathbf{x} \in S$.
- Similarly, we say that $\alpha = \sup\{f(\mathbf{x}), \mathbf{x} \in S\}$, if α is the least upper bound of f on S; that is, $\alpha \geq f(\mathbf{x})$ for all $\mathbf{x} \in S$ and there is no $\bar{\alpha} < \alpha$ such that $\bar{\alpha} \geq f(\mathbf{x})$ for all $\mathbf{x} \in S$.

Weierstrass's Theorem

Weierstrass's Theorem

Let S be a nonempty, compact set, and let $f : S \to R$ be continuous on S. Then the problem $\min\{f(\mathbf{x}) : \mathbf{x} \in S\}$ attains its minimum; that is, there exists a minimizing solution to this problem.

Weierstrass's Theorem

Proof

Here is the proof.

Separation and Support of Sets

The results of this section are based on the following geometric fact:
Given a closed convex set S and a point $\mathbf{y} \notin S$, there exists a unique point $\bar{\mathbf{x}} \in S$ with minimum distance from $\mathbf{y}$ and a hyperplane that separates $\mathbf{y}$ and S.

Closest Point Theorem

Closest Point Theorem

Let S be a nonempty, closed convex set in $\mathbb{R}^n$ and $\mathbf{y} \notin S$. Then there exists a unique point $\bar{\mathbf{x}} \in S$ with minimum distance from $\mathbf{y}$. Furthermore, $\bar{\mathbf{x}}$ is the minimizing point iif $(\mathbf{y} - \bar{\mathbf{x}})^t(\mathbf{x} - \bar{\mathbf{x}}) \leq 0$ for all $\bar{x} \in S$.

Closest Point Theorem

Proof

Here is the proof.

Hyperplanes and Seperation of Two Sets

Definition: Hyperplane and Half-Space

A hyperplane H in $\mathbb{R}^n$ is a collection of points of the form $\{\mathbf{x} : \mathbf{p}^t\mathbf{x} = \alpha\}$, where $\mathbf{p}$ is a nonzero vector in $\mathbb{R}^n$ and α is a scalar. The vector $\mathbf{p}$ is called the normal vector of the hyperplane.
A hyperplane H defines two closed half-spaces $H^+ = \{\mathbf{x} : \mathbf{p}^t\mathbf{x} \geq \alpha\}$ and $H^- = \{\mathbf{x} : \mathbf{p}^t\mathbf{x} \leq \alpha\}$, and two open half-spaces $H^+ = \{\mathbf{x} : \mathbf{p}^t\mathbf{x} > \alpha\}$ and $H^+ = \{\mathbf{x} : \mathbf{p}^t\mathbf{x} < \alpha\}$.

Hyperplanes and Seperation of Two Sets

Definition: Separation

Let S_1 and S_2 be nonempty sets in $\mathbb{R}^n$. A hyperplane $H = \{\mathbf{x} : \mathbf{p}^t\mathbf{x} = \alpha\}$, is said to separate S_1 and S_2 if $\mathbf{p}^t\mathbf{x} \geq \alpha$ for each $\mathbf{x} \in S_1$ and $\mathbf{p}^t\mathbf{x} \leq \alpha$ for each $\mathbf{x} \in S_2$.
If, in addition, $S_1 \cup S_2 \not\subset H$, H is said to properly separate S_1 and S_2.
The hyperplane H is said to strictly separate S_1 and S_2 if $\mathbf{p}^t\mathbf{x} > \alpha$ for each $\mathbf{x} \in S_1$ and $\mathbf{p}^t\mathbf{x} < \alpha$ for each $\mathbf{x} \in S_2$.
The hyperplane H is said to strongly separate S_1 and S_2 if $\mathbf{p}^t\mathbf{x} \geq \alpha + \epsilon$ for each $\mathbf{x} \in S_1$ and $\mathbf{p}^t\mathbf{x} \leq \alpha + \epsilon$ for each $\mathbf{x} \in S_2$, where ϵ is a positive scalar.

Hyperplanes and Seperation of Two Sets

Theorem: Separation Theorem

Let S be a nonempty closed convex set in $\mathbb{R}^n$ and $\mathbf{y} \notin S$. Then there exists a nonzero vector $\mathbf{p}$ and a scalar α such that $\mathbf{p}^t\mathbf{y} > \alpha$ and $\mathbf{p}^t\mathbf{x} \leq \alpha$ for each $\mathbf{x} \in S$.

Hyperplanes and Seperation of Two Sets

Proof: Separation Theorem

The set S is a nonempty closed convex set and $\mathbf{y} \notin S$. Hence, by the Closest Point Theorem, there exists a unique minimizing point $\bar{\mathbf{x}} \in S$ such that $(\mathbf{x} - \bar{\mathbf{x}})^t(\mathbf{y} - \bar{\mathbf{x}}) \leq 0$ for each $\mathbf{x} \in S$.
Letting $\mathbf{p} = \mathbf{y} - \bar{\mathbf{x}} \neq 0$ and $\alpha = \bar{\mathbf{x}}^t(\mathbf{y} - \bar{\mathbf{x}}) = \mathbf{p}^t\bar{\mathbf{x}}$, we get $\mathbf{p}^t\mathbf{x} \leq \alpha$ for each $\mathbf{x} \in S$, while

$$\mathbf{p}^t\mathbf{y} - \alpha = (\mathbf{y} - \bar{\mathbf{x}})^t(\mathbf{y} - \bar{\mathbf{x}}) = ||\mathbf{y} - \bar{\mathbf{x}}||^2 > 0$$

which completes the proof.

Hyperplanes and Seperation of Two Sets

Corollary 1

Let S be a closed convex set in $\mathbb{R}^n$. Then S is the intersection of all half-spaces containing S.

Proof

Obviously, S is contained in the intersection of all half-spaces containing it. In contradiction of the desired result, suppose that there is a point $\mathbf{y}$ in the intersection of these half-spaces but not in S. By the theorem, there exists a half-space that contains S but not $\mathbf{y}$. This contradiction proves the corollary.

Hyperplanes and Seperation of Two Sets

Corollary 2

Let S be a nonempty set, and let $\mathbf{y} \notin \text{cl}(\text{conv}(S))$, the closure of the convex hull of S. Then there exists a strongly separating hyperplane for S and $\mathbf{y}$.

Proof

The result follows by letting $\text{cl}(\text{conv}(S))$ play the role of S in the Separation Theorem.

Hyperplanes and Seperation of Two Sets

Theorem: Farkas's Theorem

Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{c}$ be an n-vector. Then exactly one of the following two systems has a solution:

System 1: $\mathbf{A}\mathbf{x} \leq \mathbf{0}$ and $\mathbf{c}^t\mathbf{x} > 0$ for some $\mathbf{x} \in \mathbb{R}^n$

System 2: $\mathbf{A}^t\mathbf{y} = \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$ for some $\mathbf{y} \in \mathbb{R}^m$.

Hyperplanes and Seperation of Two Sets

Proof: Farkas's Theorem

Suppose that System 2 has a solution; that is, there exists $\mathbf{y} \geq \mathbf{0}$ such that $\mathbf{A}^t\mathbf{y} = \mathbf{c}$. Let $\mathbf{x}$ be such that $\mathbf{Ax} \leq \mathbf{0}$. Then $\mathbf{c}^t\mathbf{x} = \mathbf{y}^t\mathbf{Ax} \leq 0$ Hence System 1 has no solution.
Now suppose that System 2 has no solution. From the set $S = \{\mathbf{x} : \mathbf{x} = \mathbf{A}^t\mathbf{y}, \mathbf{y} \geq \mathbf{0}\}$. Note that S is a closed convex set and that $\mathbf{c} \notin S$. By the Separation Theorem, there exists a vector $\mathbf{p} \in \mathbb{R}^n$ and a scalar α such that $\mathbf{p}^t\mathbf{c} > \alpha$ and $\mathbf{p}^t\mathbf{x} \leq \alpha$ for all $\mathbf{x} \in S$. Since $\mathbf{0} \in S$, $\alpha \geq 0$, so $\mathbf{p}^t\mathbf{c} > 0$. Also, $\alpha \geq \mathbf{p}^t\mathbf{A}^t\mathbf{y} = \mathbf{y}^t\mathbf{Ap}$ for all $\mathbf{y} \geq \mathbf{0}$. Since $\mathbf{y} \geq \mathbf{0}$ can be made arbitrarily large, the last inequality implies that $\mathbf{Ap} \leq \mathbf{0}$. We have therefore constructed a vector $\mathbf{p} \in \mathbb{R}^n$ such that $\mathbf{Ap} \leq \mathbf{0}$ and $\mathbf{c}^t\mathbf{p} > 0$. Hence, System 1 has a solution, and the proof is complete.

Hyperplanes and Seperation of Two Sets

Corollary 1: Gordan's Theorem

Let $\mathbf{A}$ be an $m \times n$ matrix. Then exactly one of the following two systems has a solution:

System 1: $\mathbf{Ax} \leq \mathbf{0}$ for some $\mathbf{x} \in \mathbb{R}^n$

System 2: $\mathbf{A}^t\mathbf{y} = \mathbf{0}$ and $\mathbf{y} \geq \mathbf{0}$ for some nonzero $\mathbf{y} \in \mathbb{R}^m$.

Hyperplanes and Seperation of Two Sets

Proof: Gordan's Theorem

Note that System 1 can be written equivalently as $\mathbf{A}\mathbf{x} + \mathbf{e}s \leq \mathbf{0}$ for some $\mathbf{x} \in \mathbb{R}^n$ and $s > 0$, $s \in \mathbb{R}$, where $\mathbf{e}$ is a vector of m ones. Rewriting this in the form of System 1 of Farkas's Theorem, we get

$$\begin{bmatrix} \mathbf{A} & \mathbf{e} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ s \end{bmatrix} \text{ and } \begin{pmatrix} 0 & 0 & \dots 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ s \end{pmatrix} > 0, \ \exists \begin{pmatrix} \mathbf{x} \\ s \end{pmatrix} \in \mathbb{R}^{n+1}$$

By Farkas's Theorem, the associated System 2 states that

$$\begin{bmatrix} \mathbf{A}^t \\ \mathbf{e}^t \end{bmatrix} \mathbf{y} = \begin{pmatrix} 0 & 0 & \dots 1 \end{pmatrix}^t \text{ and } \mathbf{y} \geq \mathbf{0}, \ \exists \mathbf{y} \in \mathbb{R}^m$$

that is, $\mathbf{A}^t\mathbf{y} = \mathbf{0}$, $\mathbf{e}^t\mathbf{y} = 1$, and $\mathbf{y} \geq \mathbf{0}$ for some $\mathbf{y} \in \mathbb{R}^m$. This is equivalent to System 2 of the corollary. Hence, the result follows.

Hyperplanes and Seperation of Two Sets

Corollary 2

Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{c}$ be an n-vector. Then exactly one of the following two systems has a solution:

System 1: $\mathbf{A}\mathbf{x} \leq \mathbf{0}$, $\mathbf{x} \geq 0$, $\mathbf{c}^t\mathbf{x} > 0$ for some $\mathbf{x} \in \mathbb{R}^n$

System 2: $\mathbf{A}^t\mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$ for some $\mathbf{y} \in \mathbb{R}^m$.

Hyperplanes and Seperation of Two Sets

Proof: Corollary 2

The result follows by writing the first set of constraints of System 2 as equalities and, accordingly, replacing $\mathbf{A}^t$ in the theorem by $[\mathbf{A}^t, -\mathbf{I}]$.

Hyperplanes and Seperation of Two Sets

Corollary 3

Let $\mathbf{A}$ be an $m \times n$ matrix, $\mathbf{B}$ be an $l \times n$ matrix, and $\mathbf{c}$ be an n-vector. Then exactly one of the following two systems has a solution:

System 1: $\mathbf{A}\mathbf{x} \leq \mathbf{0}$, $\mathbf{B}\mathbf{x} = \mathbf{0}$, $\mathbf{c}^t\mathbf{x} > 0$ for some $\mathbf{x} \in \mathbb{R}^n$

System 2: $\mathbf{A}^t\mathbf{y} + \mathbf{B}^t\mathbf{z} = \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$ for some $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{z} \in \mathbb{R}^l$.

Hyperplanes and Seperation of Two Sets

Proof: Corollary 3

The result follows by writing $\mathbf{z} = \mathbf{z}_1 - \mathbf{z}_2$ where $\mathbf{z_1} \geq \mathbf{0}$ and $\mathbf{z_2} \geq \mathbf{0}$ in System 2 and, accordingly, replacing $\mathbf{A}^t$ in the theorem by $[\mathbf{A}^t, \mathbf{B}^t, -\mathbf{B}^t]$.

Hyperplanes and Seperation of Two Sets

Definition: Support of Sets at Boundary Points

Let S be a nonempty set in $\mathbb{R}^n$, and let $\bar{\mathbf{x}} \in \partial S$. A hyperplane $H = \{\mathbf{x} : \mathbf{p}^t(\mathbf{x} - \bar{\mathbf{x}})\}$ is called a supporting hyperplane of S at $\bar{\mathbf{x}}$ if either $S \subseteq H^+$, that is, $\mathbf{p}^t(\mathbf{x} - \bar{\mathbf{x}}) \geq 0$ for each $\mathbf{x} \in S$, or else $S \subseteq H^-$, that is, $\mathbf{p}^t(\mathbf{x} - \bar{\mathbf{x}}) \leq 0$ for each $\mathbf{x} \in S$. If, in addition, $S \not\subseteq H$, H is called a proper supporting hyperplane of S at $\bar{\mathbf{x}}$.

Hyperplanes and Seperation of Two Sets

Theorem

Let S be a nonempty convex set in $\mathbb{R}^n$, and let $\bar{\mathbf{x}} \in \partial S$. Then there exists a hyperplane that supports S at $\bar{x}$; that is, there exists a nonzero vector $\mathbf{p}$ such that $\mathbf{p}^t(\mathbf{x} - \bar{\mathbf{x}}) \leq 0$ for each $\mathbf{x} \in \mathrm{cl}(S)$.

Hyperplanes and Seperation of Two Sets

Corollary 1

Let S be a nonempty convex set in $\mathbb{R}^n$, and let $\bar{\mathbf{x}} \notin \mathrm{int}S$. Then there is a nonzero vector $\mathbf{p}$ such that $\mathbf{p}^t(\mathbf{x} - \bar{\mathbf{x}}) \leq 0$ for each $\mathbf{x} \in \mathrm{cl}(S)$.

Hyperplanes and Seperation of Two Sets

Corollary 2

Let S be a nonempty set in $\mathbb{R}^n$, and let $\mathbf{y} \notin \text{int}(\text{conv}(S))$. Then there exists a hyperplane that separates S and $\mathbf{y}$.

Hyperplanes and Seperation of Two Sets

Corollary 3

Let S be a nonempty set in $\mathbb{R}^n$, and let $\bar{\mathbf{x}} \in \text{int}\partial S \cap \text{conv}(S)$. Then there exists a hyperplane that supports S at $\bar{\mathbf{x}}$.

Hyperplanes and Seperation of Two Sets

Theorem

Let S_1 and S_2 be nonempty convex sets in $\mathbb{R}^n$ and suppose that $S_1 \cap S_2$ is empty. Then there exists a hyperplane that separates S_1 and S_2; that is, there exists a nonzero vector $\mathbf{p} \in \mathbb{R}^n$ such that

$$\inf\{\mathbf{p}^t\mathbf{x} : \mathbf{x} \in S_1\} \geq \sup\{\mathbf{p}^t\mathbf{x} : \mathbf{x} \in S_2\}$$

Hyperplanes and Seperation of Two Sets

Theorem: Gordan's Theorem

Let $\mathbf{A}$ be an $m \times n$ matrix. Then exactly one of the following two systems has a solution:

System 1: $\mathbf{A}\mathbf{x} < \mathbf{0}$ for some $\mathbf{x} \in \mathbb{R}^n$

System 2: $\mathbf{A}^t\mathbf{p} = \mathbf{0}$ and $\mathbf{p} \geq \mathbf{0}$ for some nonzero $\mathbf{p} \in \mathbb{R}^m$.

Hyperplanes and Seperation of Two Sets

Theorem: Strong Separation Theorem

Let S_1 and S_2 be closed convex sets, and suppose that S_1 is bounded. If $S_1 \cap S_2$ is empty, there exists a hyperplane that strongly separates S_1 and S_2; that is, there exists a nonzero vector $\mathbf{p}$ and $\epsilon > 0$ such that

$$\inf\{\mathbf{p}^t\mathbf{x} : \mathbf{x} \in S_1\} \geq \epsilon + \sup\{\mathbf{p}^t\mathbf{x} : \mathbf{x} \in S_2\}$$

Convex Cones and Polarity

Definition: Convex Cone

A nonempty set C in $\mathbb{R}^n$ is called a cone with vertex zero if $\mathbf{x} \in C$ implies that $\lambda\mathbf{x} \in C$ for all $\lambda \geq 0$. If, in addition, C is convex, C is called a convex cone.

Convex Cones and Polarity

An important special class of convex cones is that of polar cones.

Definition: Polar Cone

Let S be a nonempty set $\mathbb{R}^n$. Then the polar cone of S, denoted S^*, is given by $\{\mathbf{p} : \mathbf{p}^t\mathbf{x} \leq 0, \forall \mathbf{x} \in S\}$. If S is empty, S^* will be interpreted as $\mathbb{R}^n$.

Convex Cones and Polarity

Definition: Polar Cone

Let S, S_1, and S_2 be nonempty sets in $\mathbb{R}^n$. Then the following statements hold true.

1. S^* is a closed convex cone.
2. $S \subseteq S^{**}$, where S^{**} is the polar cone of S^*.
3. $S_1 \subseteq S_2$ implies that $S_2^* \subseteq S_1^*$.

Convex Cones and Polarity

Theorem

Let C be a nonempty closed convex cone. Then $C = C^{**}$.

Polyhedral Sets

Definition: Polyhedral Sets

A set in $\mathbb{R}^n$ is called a polyhedral set if it is the intersection of a finite number of closed half-spaces; that is,
$\{\mathbf{p} : \mathbf{p}_i^t \mathbf{x} \leq \alpha_i, \forall i = 1, \ldots m\}$, where $\mathbf{p}_i$ is a nonzero vector and α_i is a scalar for for $i = 1, \ldots m$.

Extreme Points

Definition: Extreme Points

Let S be nonempty convex set in $\mathbb{R}^n$. A vector $\mathbf{x}$ is called an extreme point of S if $\mathbf{x} = \lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$ with $\mathbf{x}_1, \mathbf{x}_2 \in S$, and $\lambda \in (0,1)$ implies that $\mathbf{x} = \mathbf{x_1} = \mathbf{x_2}$.

Extreme Directions

Definition: Extreme Directions

Let S be a nonempty, closed convex set in $\mathbb{R}^n$. A nonzero vector $\mathbf{d}$ in $\mathbf{R}^n$ is called a direction, or a recession direction, of S if for each $\mathbf{x} \in S$, $\mathbf{x} + \lambda\mathbf{d} \in S$ for all $\lambda \geq 0$. Two directions $\mathbf{d}_1$ and $\mathbf{d}_2$ of S are called distinct if $\mathbf{d}_1 \neq \alpha\mathbf{d}_2$ for any $\alpha > 0$.

A direction $\mathbf{d}$ of S is called an extreme direction if it cannot be written as a positive linear combination of two distinct directions; that is, if $\mathbf{d} = \lambda_1\mathbf{d}_1 + \lambda_2\mathbf{d}_2$ for $\lambda_1, \lambda_2 > 0$, then $\mathbf{d}_1 = \alpha\mathbf{d}_2$ for some $\alpha > 0$.

Characterization of Extreme Points

Theorem: Characterization of Extreme Points

Let $S = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, where $\mathbf{A}$ is an $m \times n$ matrix of rank m and $\mathbf{b}$ is an m-vector. A point $\mathbf{x}$ is an extreme point of S if and only if $\mathbf{A}$ can be decomposed into $[\mathbf{B}, \mathbf{N}]$ such that

$$\begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix} = \begin{bmatrix} \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} \end{bmatrix}$$

where $\mathbf{B}$ is an $m \times m$ invertible matrix satisfying $\mathbf{B}^{-1}\mathbf{b} \geq \mathbf{0}$. Any such solution is called a basic feasible solution (BFS) for S.

Characterization of Extreme Points

Corollary

The number of extreme points of S is finite.

Characterization of Extreme Points

Theorem: Existence of Extreme Points

Let $S = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ be nonempty, where $\mathbf{A}$ is an $m \times n$ matrix of rank m and b is an m-vector. Then S has at least one extreme point.

Characterization of Extreme Directions

Theorem: Characterization of Extreme Directions

Let $S = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ be nonempty, where $\mathbf{A}$ is an $m \times n$ matrix of rank m and b is an m-vector. A vector $\bar{\mathbf{d}}$ is an extreme direction of S if and only if $\mathbf{A}$ can be decomposed into $[\mathbf{B}, \mathbf{N}]$ such that $\mathbf{B}^{-1}\mathbf{a}_j \leq \mathbf{0}$ for some column $\mathbf{a}_j$ of $\mathbf{N}$, and $\bar{\mathbf{d}}$ is a positive multiple of $\mathbf{d}$,

$$\mathbf{d} = \begin{pmatrix} -\mathbf{B}^{-1}\mathbf{a}_j \\ \mathbf{e}_j \end{pmatrix}$$

where e_j is an $n - m$ elementary vector.

Characterization of Extreme Directions

Corollary

The number of extreme directions of S is finite.

Representation Theorem

Theorem: Representation Theorem

Let $S = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ be nonempty, where $\mathbf{A}$ is an $m \times n$ matrix of rank m and b is an m-vector. Let $\mathbf{x}_1, \ldots, \mathbf{x}_k$ be the extreme points of S and $\mathbf{d}_1, \ldots, \mathbf{d}_l$ be the extreme directions of S. Then $\mathbf{x} \in S$ if and only if $\mathbf{x}$ can be written as

$$\mathbf{x} = \sum_{j=1}^{k} \lambda_j \mathbf{x}_j + \sum_{j=1}^{l} \mu_j \mathbf{d}_j$$

$$\sum_{j=1}^{k} \lambda_j = 1$$

where $\lambda_j \geq 0, \forall j$ and $\mu_j \geq 0, \forall j$.

Representation Theorem

Corollary: Existence of Extreme Directions

Let $S = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ be nonempty, where $\mathbf{A}$ is an $m \times n$ matrix of rank m and b is an m-vector. Then S has at least one extreme direction if and only if it is unbounded.

Linear Programming and the Simplex Method

Please see Linear Programming lecture notes for a review of Linear Programming and the Simplex Method.

Summary

- Convex Hulls
- Closure and Interior of a Set
- Weierstrass's Theorem
- Separation and Support of Sets
- Convex Cones and Polarity
- Polyhedral Sets, Extreme Points and Extreme Directions
- Linear Programming and the Simplex Method

Thanks! Questions?