Introduction to Nonlinear Programming

Nonlinear Programming

Fatih Cavdur

to accompany
Nonlinear Programming
by Mokhtar S. Bazara, Hanif D. Sherali and C. M. Shetty

Problem Statement and Basic Definitions

We want to minimize or maximize the function

$$f(\mathbf{x})$$

subject to constraints

$$g_i(\mathbf{x}) \leq 0,\ \forall i$$

$$h_i(\mathbf{x}) = 0,\ \forall i$$

$$\mathbf{x} \in \mathbf{X}$$

Problem Statement and Basic Definitions

Example

We want to minimize

$$f(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 2)^2$$

subject to constraints

$$g_1(x_1, x_2) = x_1^2 - x_2 - 3 \leq 0$$

$$g_2(x_1, x_2) = x_2 - 1 \leq 0$$

$$g_3(x_1, x_2) = -x_1 \leq 0$$

Illustrate the problem solution!

Discrete Optimal Control

We want to minimize

$$\alpha(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_K, \mathbf{u}_1, \ldots, \mathbf{u}_K)$$

subject to constraints

$$\mathbf{y}_k = \mathbf{y}_{k-1} + \phi(\mathbf{y}_{k-1}, \mathbf{u}_k),\ \forall k$$

$$\mathbf{y}_k \in \mathbf{Y}_k \text{ and } \mathbf{u}_k \in \mathbf{U}_k,\ \forall k$$

$$\Psi(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_K, \mathbf{u}_1, \ldots, \mathbf{u}_K) \in \mathbf{D}$$

Discrete Optimal Control

Example

We want to minimize

$$\sum_{k=1}^{K} (c_1 u_k^2 + c_2 I_k)^2$$

subject to constraints

$$L_k = L_{k-1} + u_k,\ \forall k$$

$$I_k = I_{k-1} + pL_{k-1} - d_k,\ \forall k$$

$$0 \leq L_k \leq b/p \text{ and } I_k \geq 0,\ \forall k$$

Continuous Optimal Control

We want to minimize

$$\int_0^T \alpha[\mathbf{y}(t), \mathbf{u}(t)]dt$$

subject to constraints

$$\dot{\mathbf{y}}(t) = \Phi[\mathbf{y}(t), \mathbf{u}(t)],\ \ t \in [0, T]$$

$$\mathbf{y}_k \in \mathbf{Y}_k \ \mathit{and}\ \mathbf{u}_k \in \mathbf{U}_k,\ \ \forall k$$

$$\Psi(\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_K, \mathbf{u}_1, \ldots, \mathbf{u}_K) \in \mathbf{D}$$

Continuous Optimal Control

Example

We want to minimize

$$\int_0^T |u(t)|\dot{y}(t)dt$$

subject to constraints

$$m\ddot{y}(t) + m(g) = u(t),\ \ t \in [0, T]$$

$$|u(t)| \leq b,\ \ t \in [0, T]$$

$$y(T) = \bar{y}$$

Continuous Optimal Control

Example

We want to minimize

$$\int_0^T |y(t) - c(t)| dt$$

subject to constraints

$$|\dot{y}(t)| \leq b_1,\ \ t \in [0, T]$$

$$|\ddot{y}(t)| \leq b_2,\ \ t \in [0, T]$$

$$y(0) = a \text{ and } y(T) = b$$

Stochastic Resource Allocation

Consider the constraints

$$\mathbf{Ax} \leq \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

with the following objectives

$$\max\ \bar{\mathbf{c}}^t\mathbf{x}\ \Rightarrow\ \text{LP}$$

$$\min\ \mathbf{x}^t\mathbf{V}\mathbf{x}\ \Rightarrow\ \text{QP}$$

Stochastic Resource Allocation

Satisficing Criteria

We want to minimize

$$\min z = \mathbf{x}^t \mathbf{V} \mathbf{x}$$

subject to constraints

$$\mathbf{A}\mathbf{x} \leq \mathbf{b}$$

$$\bar{\mathbf{c}}^t \mathbf{x} \geq \bar{z}$$

$$\mathbf{x} \geq \mathbf{0}$$

Stochastic Resource Allocation

Satisficing Criteria

We want to minimize

$$\min z = \mathbf{x}^t \mathbf{V} \mathbf{x}$$

subject to constraints

$$\mathbf{A}\mathbf{x} \leq \mathbf{b}$$

$$\bar{\mathbf{c}}^t \mathbf{x} \geq \bar{z}$$

$$\mathbf{x} \geq \mathbf{0}$$

Stochastic Resource Allocation

Another Approach

Let α be defined as

$$\alpha = P\{c^t\mathbf{x} \geq \bar{z}\}$$

Now, suppose the vector of RVs $\mathbf{c}$ can be expressed as

$$\mathbf{c} = \mathbf{d} + y\mathbf{f}$$

We then have

$$\alpha = P\{\mathbf{d}^t\mathbf{x} + y\mathbf{f}^t\mathbf{x} \geq \bar{z}\} = P\left\{y \geq \frac{\bar{z} - \mathbf{d}^t\mathbf{x}}{\mathbf{f}^t\mathbf{x}}\right\}$$

Stochastic Resource Allocation

Satisficing Criteria

We then have the following Linear Fractional Programming problem

$$\min z = \frac{\bar{z} - \mathbf{d}^t\mathbf{x}}{\mathbf{f}^t\mathbf{x}}$$

subject to constraints

$$\mathbf{A}\mathbf{x} \leq \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

Stochastic Resource Allocation

Satisficing Criteria

If we also want to include a constraint that required the probability of the profit exceeding the desired value to be at least some specified value q, we can use the following chance constraint

$$\alpha = P\{c^t\mathbf{x} \geq \bar{z}\} = P\left\{y \geq \frac{\bar{z} - \mathbf{d}^t\mathbf{x}}{\mathbf{f}^t\mathbf{x}}\right\} \geq q$$

If we assume that y is continuous RV for which ϕ_q is the upper $100q$ percentile ($P\{y \geq \phi_q\} = q$), we can write it as

$$\frac{\bar{z} - \mathbf{d}^t\mathbf{x}}{\mathbf{f}^t\mathbf{x}} \leq \phi_q \Rightarrow \mathbf{d}^t\mathbf{x} + \phi_q\mathbf{f}^t\mathbf{x} \geq \bar{z}$$

Stochastic Resource Allocation

Risk Aversion

We can define a utility function as

$$u(z) = 1 - e^{-kz}$$

where $k > 0$ is a risk aversion constant. Note that increasing k results in a more risk-averse behavior.
Suppose that the current worth is zero, so that the total worth is equal to the gain z. Also suppose that $\mathbf{c}$ is a normal random vector with mean $\bar{\mathbf{c}}$ and covariance matrix $\mathbf{V}$. z is then a normal RV with mean $\bar{z} = \bar{\mathbf{c}}^t\mathbf{x}$ and variance $\sigma^2 = \bar{\mathbf{c}}^t\mathbf{V}\mathbf{x}$. In particular,

$$\phi(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\bar{z})^2}{2\sigma^2}}$$

Stochastic Resource Allocation

Risk Aversion

We want to maximize the expected utility as

$$
\begin{aligned}
E(U) &= \int_{-\infty}^{+\infty} (1 - e^{-kz})\phi(z)dz \\
&= 1 - \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-kz - \frac{(z-\bar{z})^2}{2\sigma^2}} \\
&= 1 - \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{z-\bar{z}+k\sigma^2}{\sigma}\right)^2} e^{\left(-k\bar{z}\frac{1}{2}k^2\sigma^2\right)} dz \\
&= 1 - e^{\frac{\left(-k\bar{z}\frac{1}{2}k^2\sigma^2\right)}{\sqrt{2\pi\sigma^2}}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{z-\bar{z}+k\sigma^2}{\sigma}\right)^2} dz \\
&= 1 - e^{\left(-k\bar{z}\frac{1}{2}k^2\sigma^2\right)}
\end{aligned}
$$

Location of Facilities

$$
\min z = \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij1} d_{ij}
$$

subject to constraints

$$
\sum_{j=1}^{n} w_{ij} \leq c_i, \ \forall i
$$

$$
\sum_{i=1}^{m} w_{ij} \leq r_j, \ \forall j
$$

$$
w_{ij} \geq 0, \ \forall i, j
$$

Location of Facilities

Consideration of different norms, such as L_1, L_2 or L_p, we have different nonlinear programming models as

$$d_{ij} = |x_i - a_j| + |y_i - b_j|$$

$$d_{ij} = \left[(x_i - a_j)^2 + y_i - b_j)^2\right]^{1/2}$$

$$d_{ij} = \left[(x_i - a_j)^2 + y_i - b_j)^2\right]^{1/p}$$

Guidelines for Model Construction

- Mathematical abstraction of a given problem
- Identification or formulation of the problem
- A balanced compromise between representability and mathematical tractability
- Often several ways of construction of a mathematical statement of the problem
- Although mathematically equivalent, substantial difference in felicity these alternatives afford to solution algorithms

Guidelines for Model Construction

Example

For example, we can write the inequality and equality constraints of the NLP in the previous slides as

$$\sum_{i=1}^{m} [g_i(\mathbf{x}) + s_i^2]^2 + \sum_{j=1}^{l} [h_j^2(\mathbf{x}) = 0$$

$$\sum_{i=1}^{m} \max\{g_i(\mathbf{x}), 0\} + \sum_{j=1}^{l} |h_j(\mathbf{x})| = 0$$

$$\sum_{i=1}^{m} \max^2\{g_i(\mathbf{x}), 0\} + \sum_{j=1}^{l} h_j^2(\mathbf{x}) = 0$$

Guidelines for Model Construction

Guidelines

- Special Structures
- Complex Structures
- Preemptive Priority Strategy
- Hard-Soft or Elastic Constraints
- Bounding and Scaling

Summary

- Problem Statement and Basic Definitions
- Illustrative Examples
- Guidelines for Model Construction

Thanks! Questions?