Probability and Statistics
Lecture 6: Some Continuous Distributions

to accompany

Probability and Statistics for Engineers and Scientists

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Uniform Distribution

PDF of the uniform distribution is given by

$$f(x;A,B)=\begin{cases}\dfrac{1}{B-A}, & A\le x\le B\\ 0, & \text{otherwise}\end{cases}$$

The mean and variance of the uniform distribution is given by

$$\mu=\frac{A+B}{2} \quad \text{and} \quad \sigma^2=\frac{(B-A)^2}{12}$$

Uniform Distribution

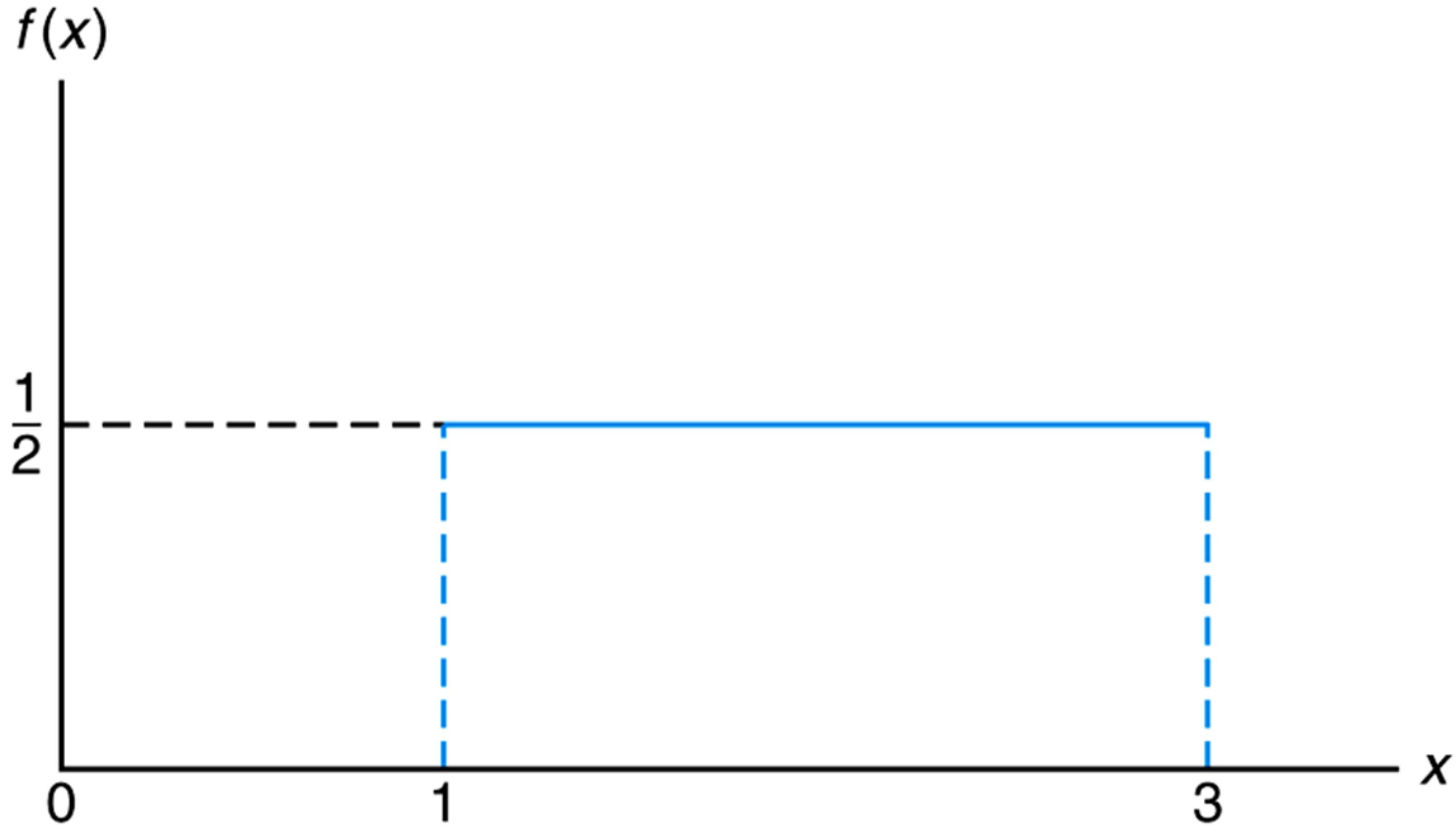
Find the probability that $P\{X \geq 3\}$, if we have

$$f(x;0,4) = \begin{cases} \frac{1}{4}, & 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

We have

$$P\{X \geq 3\} = \int_3^4 \frac{dx}{4} = \frac{1}{4}$$

Uniform Distribution



Uniform Distribution

The mean and variance of the uniform distribution are

$$\mu = \frac{A+B}{2} \text{ and } \sigma^2 = \frac{(B-A)^2}{12}.$$

Normal Distribution

PDF of the normal distribution with mean μ and variance σ^2 is given by

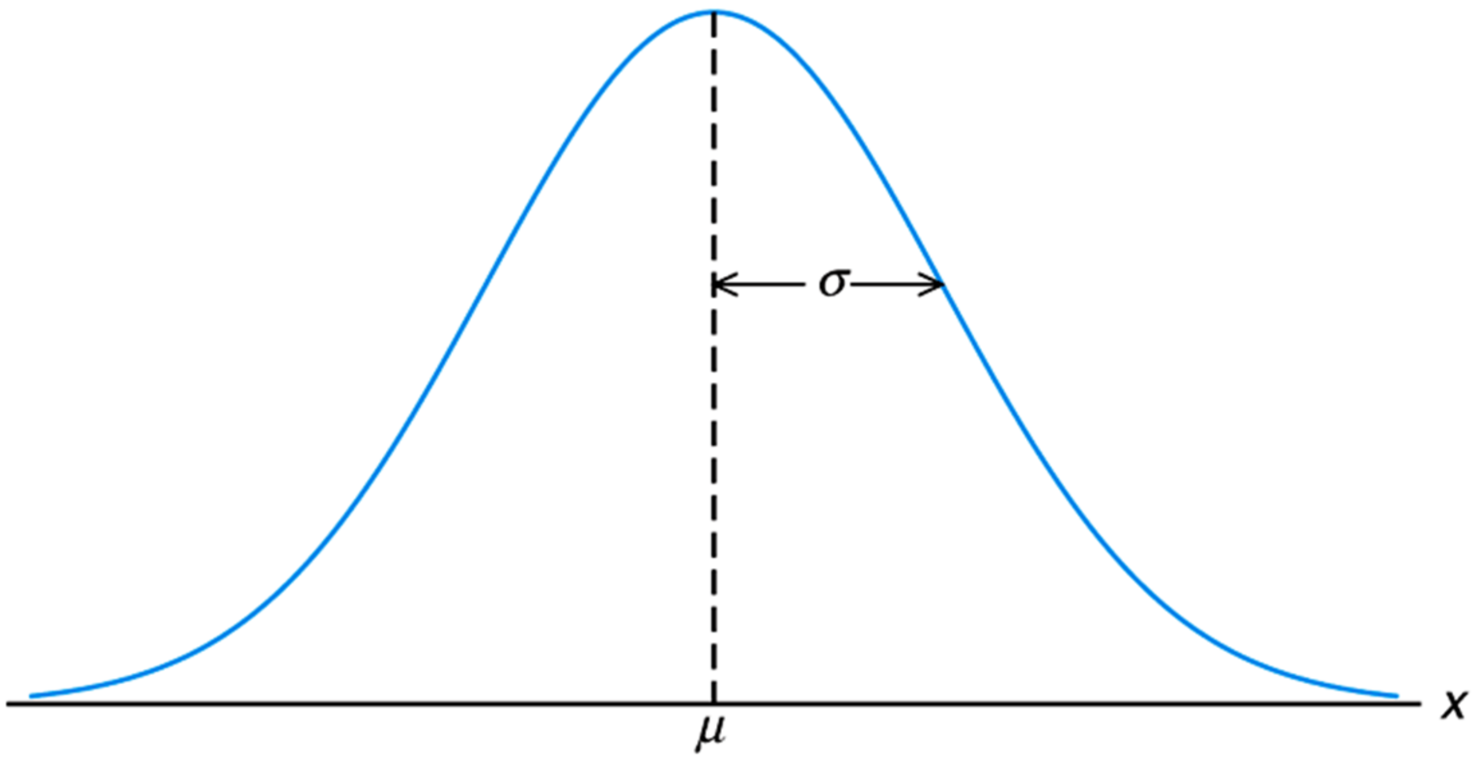
$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

for $-\infty < x < +\infty$.

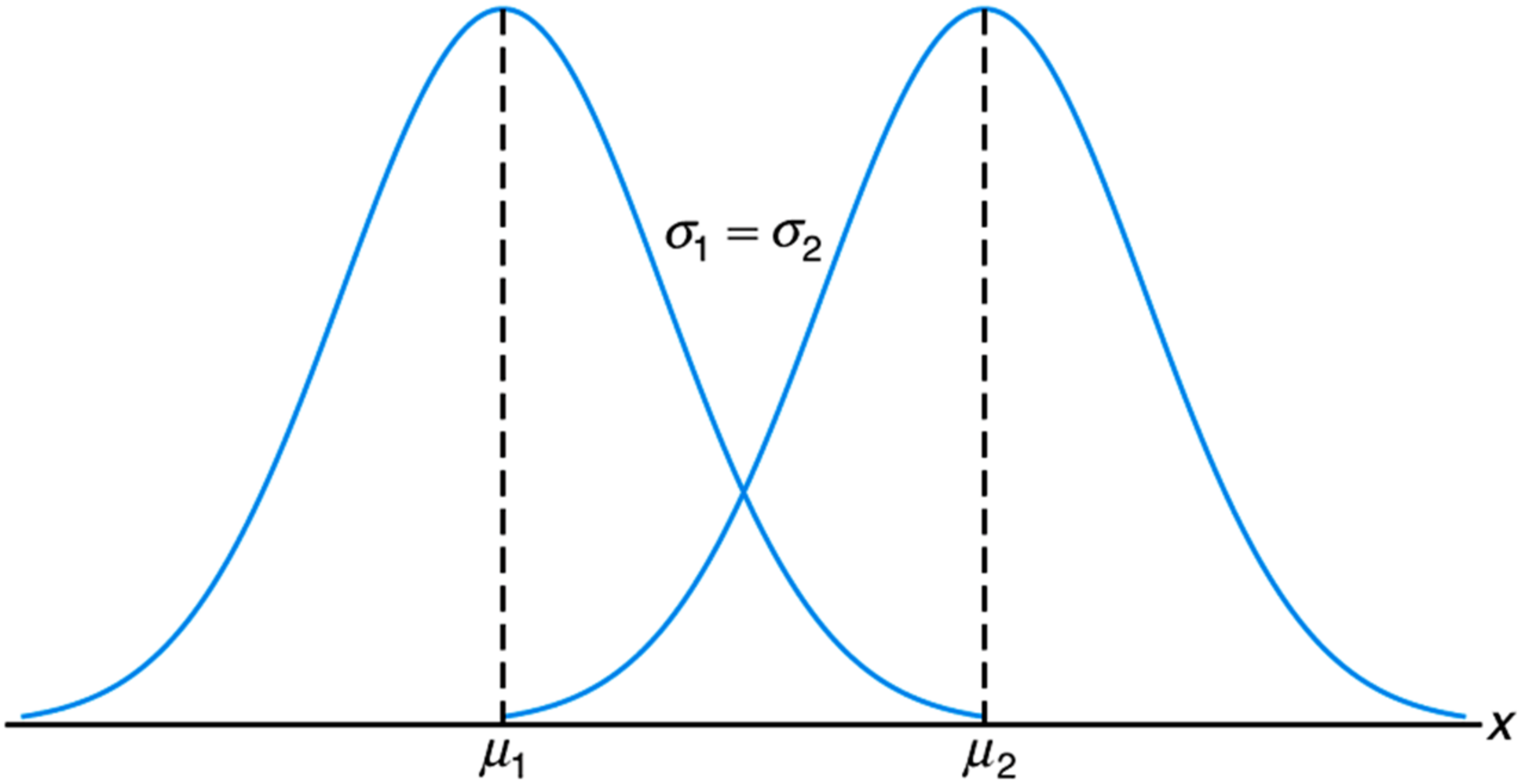
The mean and variance of the normal distribution is given by

$$\mu \text{ and } \sigma^2$$

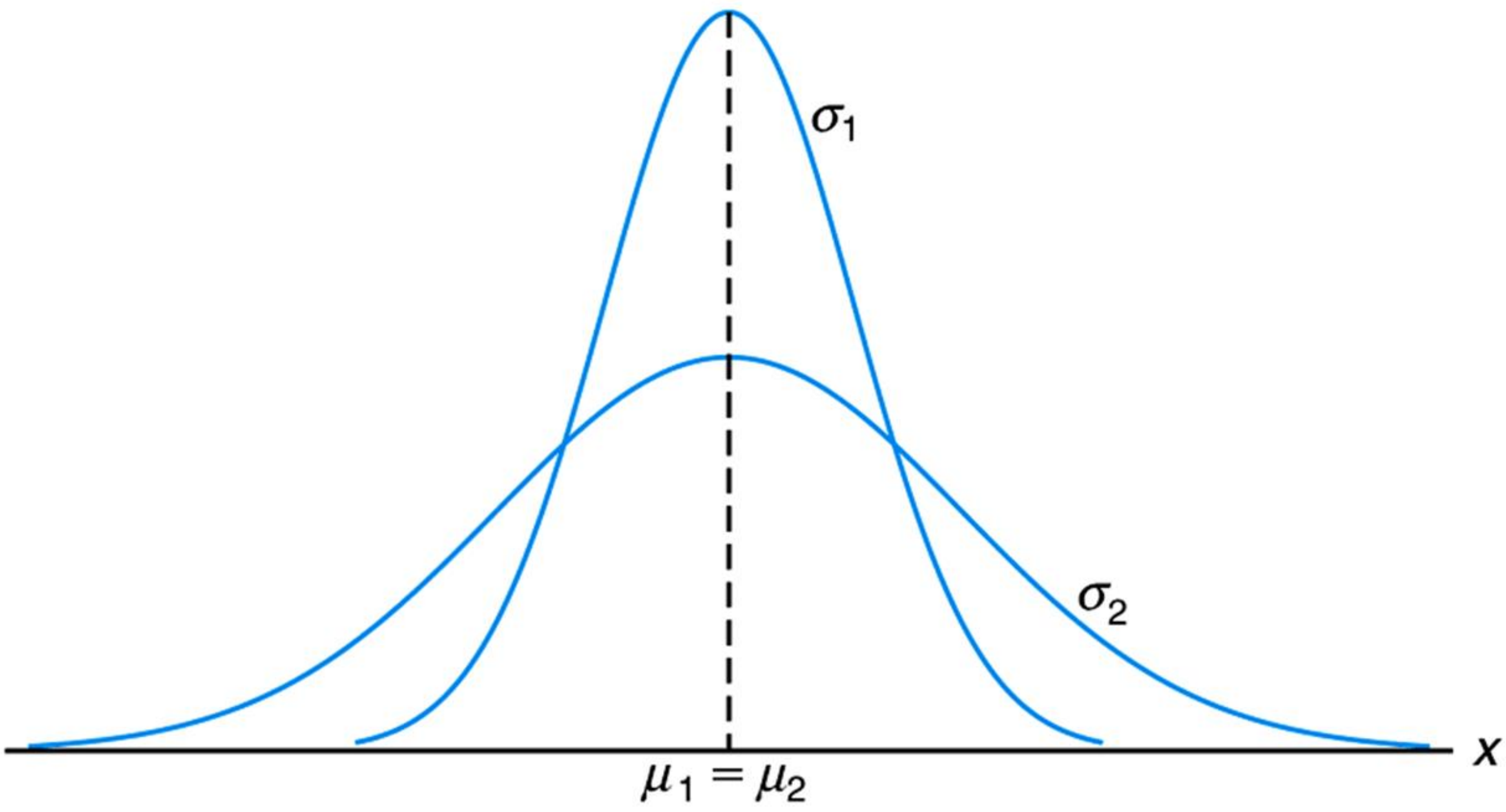
Normal Distribution



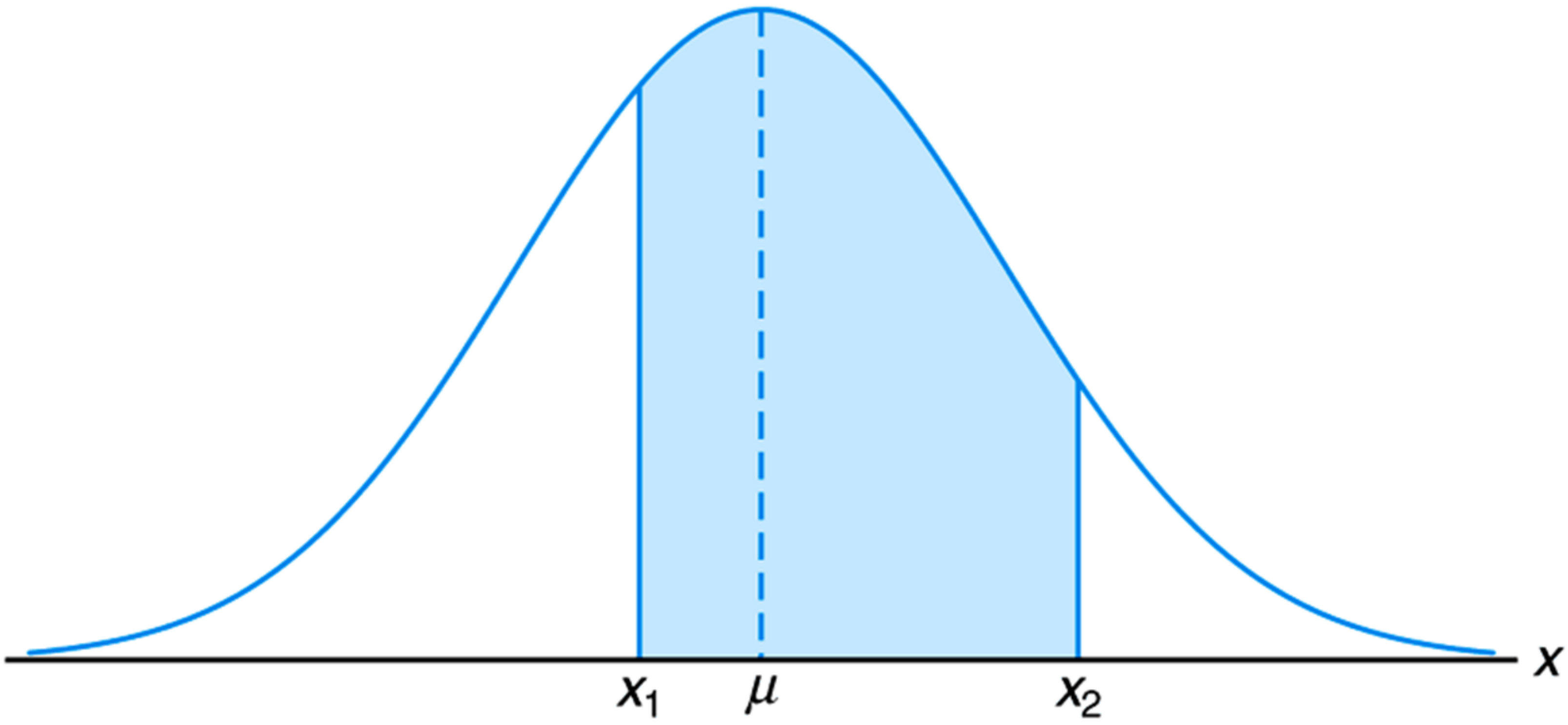
Normal Distribution



Normal Distribution



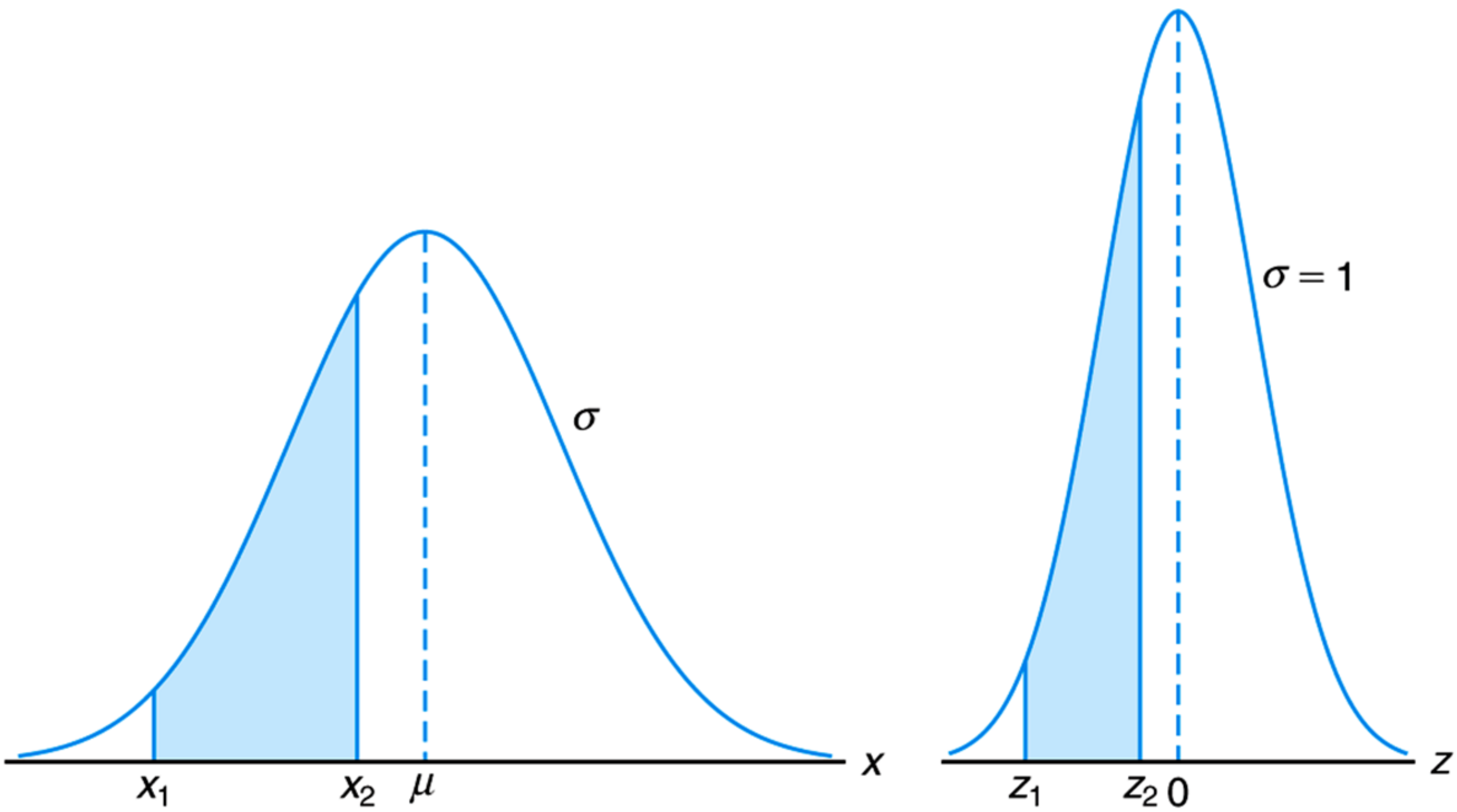
Normal Distribution



Standard Normal Distribution

The distribution of a normal random variable with mean 0 and variance 1 is called a **standard normal distribution**.

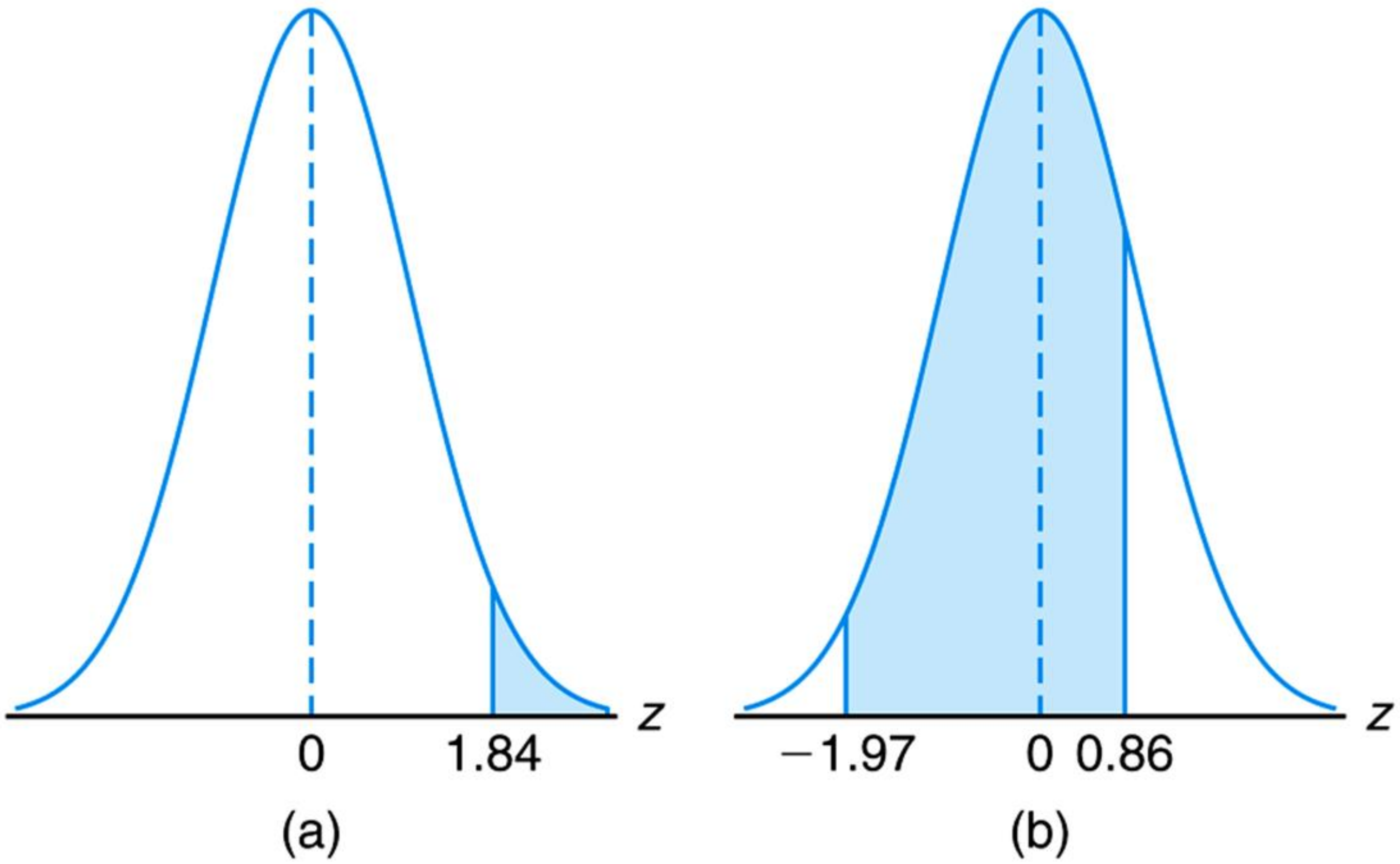
Standard Normal Distribution



Example

Example 6.2: For a standard normal distribution, find the area under the curve that lies to the right of $z = 1.84$ and between $z = -1.97$ and $z = 0.86$.

Example



Example

The area under the curve that lies to the right of $z = 1.84$ is

$$P\{Z \leq 1.84\} = 1 - \phi(1.84) = 1 - 0.9671 = 0.0329$$

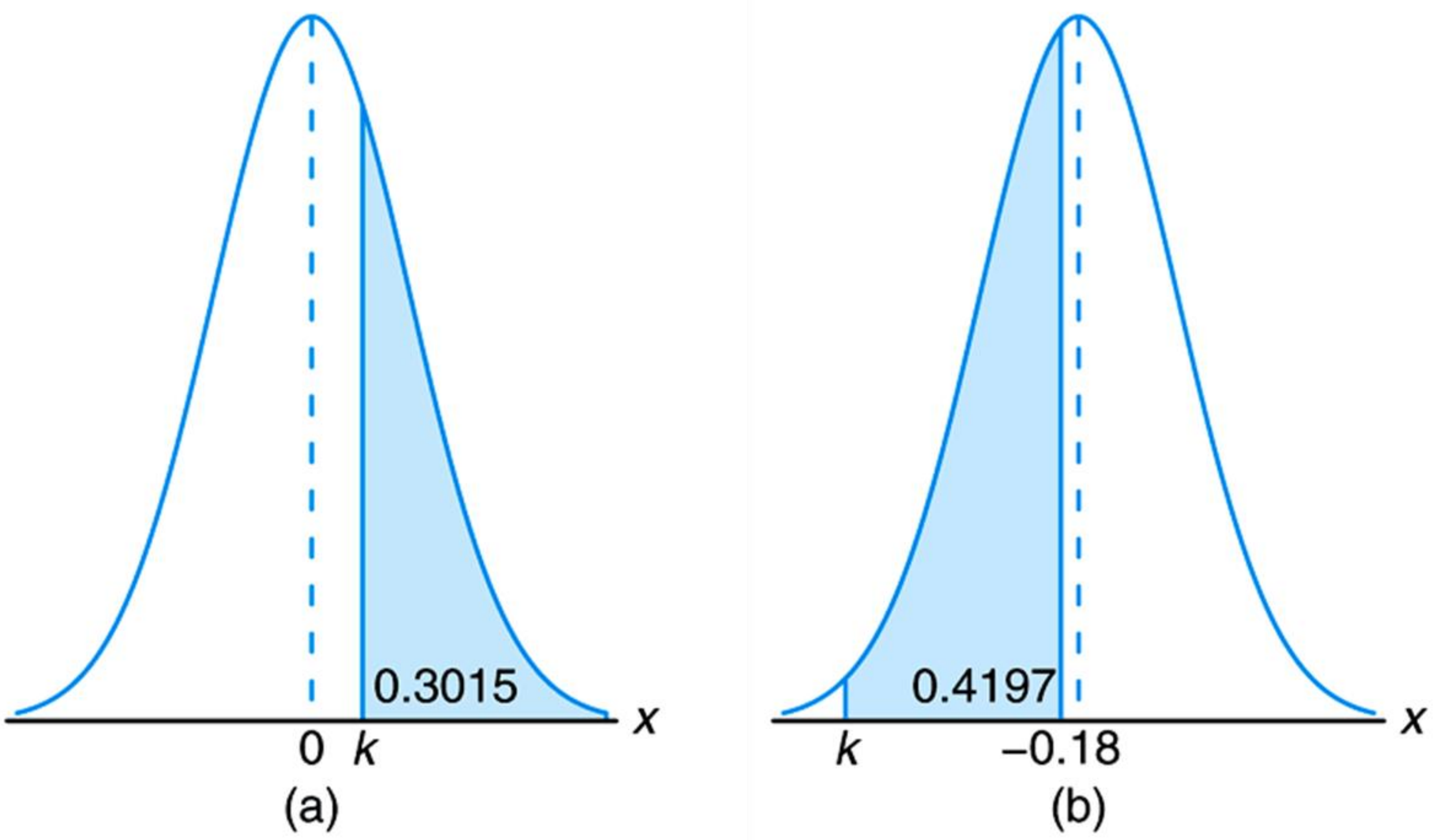
Similarly, the area under the curve that lies between $z = -1.97$ and $z = 0.86$ is

$$\begin{aligned} P\{-1.97 \leq Z \leq 0.86\} &= \phi(0.86) - \phi(-1.97) \\ &= 0.8051 - 0.0244 \\ &= 0.7807 \end{aligned}$$

Example

Example 6.3: For a standard normal distribution, find the value of k such that $P\{Z > k\} = 0.3015$ and $P\{k < Z < -0.18\} = 0.4197$.

Example



Example

We have that

$$P\{Z > k\} = 0.3015 = 1 - P\{Z \leq k\} \Rightarrow \phi(k) = 0.6985 \Rightarrow k = 0.52$$

Similarly,

$$\begin{aligned} P\{k < Z < -0.18\} &= 0.4197 \\ &= \phi(-0.18) - \phi(k) \\ &\Rightarrow \phi(k) = 0.0089 \\ &\Rightarrow k = -2.37 \end{aligned}$$

Example

Example 6.4: Given an RV X with normal distribution with $\mu = 50$ and $\sigma = 10$, find the probability that X is between 45 and 62.

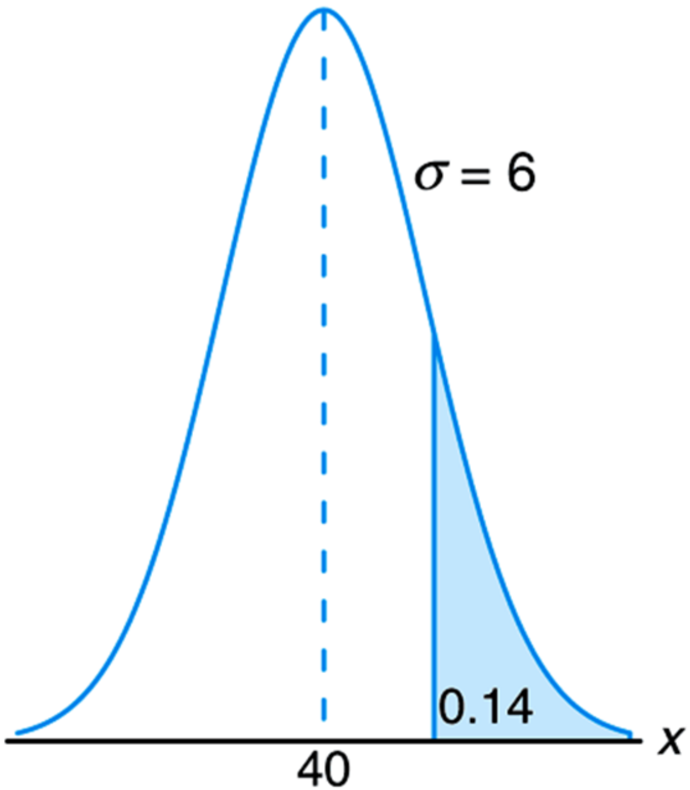
We can write

$$\begin{aligned} P\{45 < X < 62\} &= P\left\{\frac{45-50}{10} < Z < \frac{62-50}{10}\right\} \\ &= P\{-0.5 < Z < 1.2\} \\ &= \phi(1.2) - \phi(-0.5) \\ &= 0.8840 - 0.3085 \\ &= 0.5764 \end{aligned}$$

Example

Example 6.6: For a normal RV X with $\mu = 40$ and $\sigma = 6$, find the value of x that has 14% of the area to the right.

Example



(b)

Example

We have

$$1 - \phi(k) = 0.14 \Rightarrow \phi(k) = 0.86 \Rightarrow k = 1.08$$

$$k = 1.08 = \frac{x - 40}{6} \Rightarrow x = 46.48$$

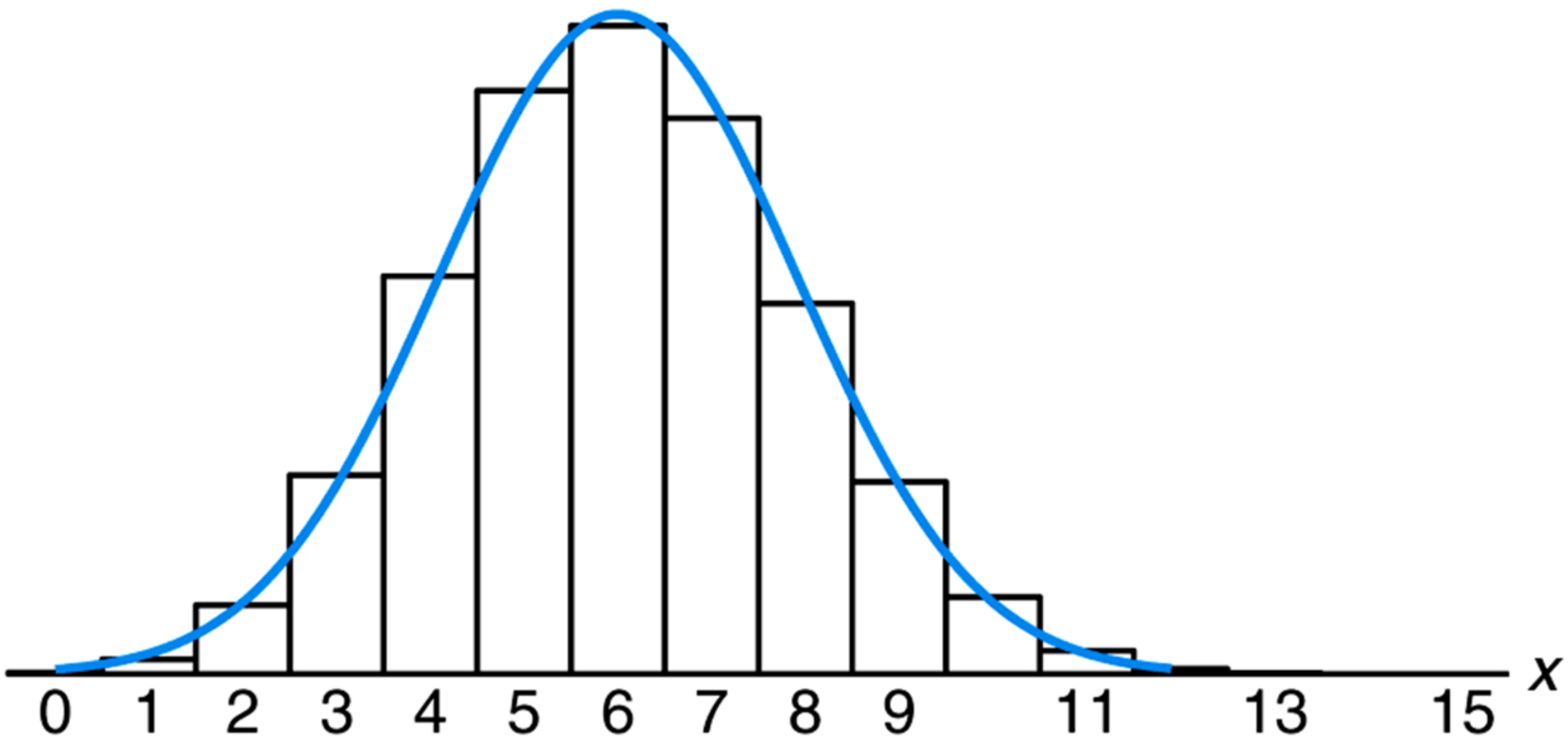
Normal Approximation to Binomial

If X is a binomial random variable with mean $\mu = np$ and variance $\sigma^2 = npq$, then the limiting form of the distribution of

$$Z = \frac{X - np}{\sqrt{npq}},$$

as $n \to \infty$, is the standard normal distribution $n(z; 0, 1)$.

Normal Approximation to Binomial



Normal Approximation to Binomial

	$p = 0.05,\ n = 10$		$p = 0.10,\ n = 10$		$p = 0.50,\ n = 10$	
r	**Binomial**	**Normal**	**Binomial**	**Normal**	**Binomial**	**Normal**
0	0.5987	0.5000	0.3487	0.2981	0.0010	0.0022
1	0.9139	0.9265	0.7361	0.7019	0.0107	0.0136
2	0.9885	0.9981	0.9298	0.9429	0.0547	0.0571
3	0.9990	1.0000	0.9872	0.9959	0.1719	0.1711
4	1.0000	1.0000	0.9984	0.9999	0.3770	0.3745
5			1.0000	1.0000	0.6230	0.6255
6					0.8281	0.8289
7					0.9453	0.9429
8					0.9893	0.9864
9					0.9990	0.9978
10					1.0000	0.9997

	$p = 0.05$					
	$n = 20$		$n = 50$		$n = 100$	
r	**Binomial**	**Normal**	**Binomial**	**Normal**	**Binomial**	**Normal**
0	0.3585	0.3015	0.0769	0.0968	0.0059	0.0197
1	0.7358	0.6985	0.2794	0.2578	0.0371	0.0537
2	0.9245	0.9382	0.5405	0.5000	0.1183	0.1251
3	0.9841	0.9948	0.7604	0.7422	0.2578	0.2451
4	0.9974	0.9998	0.8964	0.9032	0.4360	0.4090
5	0.9997	1.0000	0.9622	0.9744	0.6160	0.5910
6	1.0000	1.0000	0.9882	0.9953	0.7660	0.7549
7			0.9968	0.9994	0.8720	0.8749
8			0.9992	0.9999	0.9369	0.9463
9			0.9998	1.0000	0.9718	0.9803
10			1.0000	1.0000	0.9885	0.9941

Gamma and Exponential Distribution

PDF of the gamma distribution is given by

$$f(x;\alpha,\beta) = \begin{cases} \dfrac{x^{\alpha-1}e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$ and Γ is the gamma function defined as

$$\Gamma(\alpha) = \int_{0}^{\infty} x^{\alpha-1}e^{-x}dx$$

where $\alpha > 0$.

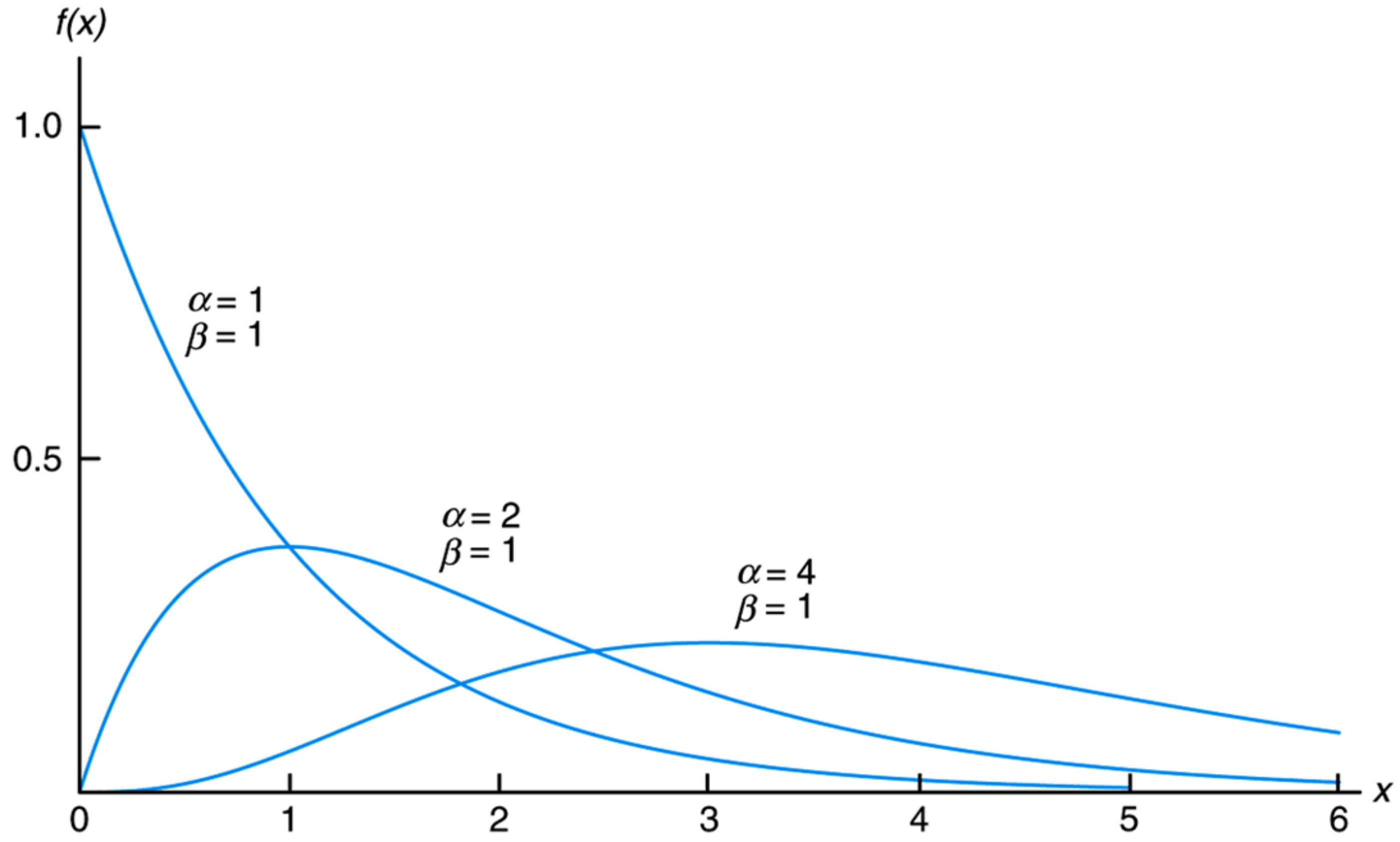
Gamma and Exponential Distribution

PDF of the exponential distribution is given by

$$f(x;\alpha,\beta) = \begin{cases} \dfrac{e^{-x/\beta}}{\beta}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

where $\beta > 0$.

Gamma and Exponential Distribution



Gamma and Exponential Distribution

Example 6.17: A system contains a certain type of component whose time to failure is given by the RV T (in years). The RV is modeled nicely by the exponential distribution with mean time to failure is 5 years. If 5 of these components are installed in different systems, what is the probability that at least 2 are still functioning at the end of 8 years?

Gamma and Exponential Distribution

We have

$$P\{T > 8\} = \frac{1}{5}\int_{8}^{\infty} e^{-t/5}dt = e^{-8/5} \cong 0.2$$

Gamma and Exponential Distribution

If we let X be the number of components functioning at the end of 8 years, then, X is binomial with $n = 5$ and $p = 0.2$.

$$
\begin{aligned}
P\{X \geq 2\} &= 1 - P\{X \leq 1\} \\
&= 1 - \sum_{x=0}^{1} \binom{5}{x} (0.2)^x (1 - 0.2)^{5-x} \\
&= 0.2627
\end{aligned}
$$

Gamma and Exponential Distribution

The mean and variance of the gamma distribution are

$$\mu = \alpha\beta \text{ and } \sigma^2 = \alpha\beta^2$$

The mean and variance of the exponential distribution are

$$\mu = \beta \text{ and } \sigma^2 = \beta^2$$

Chi-Square Distribution

PDF of the chi-squared distribution is given by

$$f(x;v) = \begin{cases} \dfrac{x^{\frac{v}{2}-1}e^{-\frac{x}{2}}}{2^{v/2}\Gamma\left(\frac{v}{2}\right)}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

where the degrees of freedom $v \in \mathbb{Z}^{+}$.

The mean and the variance of the chi-squared distribution are

$$\mu = v \text{ and } \sigma^2 = 2v$$

Beta Distribution

PDF of the beta distribution is

$$f(x;\alpha,\beta) = \begin{cases} \dfrac{x^{\alpha-1}(1-x)^{\beta-1}}{\mathrm{B}(\alpha,\beta)}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$, and B is the beta function defined as

$$\mathrm{B}(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

where $\alpha > 0$ and $\beta > 0$

Beta Distribution

The mean and the variance of the beta distribution are

$$\mu = \frac{\alpha}{\alpha + \beta}$$

and

$$\sigma^2 = \frac{\alpha\beta}{(\alpha + \beta)(\alpha + \beta + 1)}$$

Log-Normal Distribution

PDF of the log-normal distribution is given by

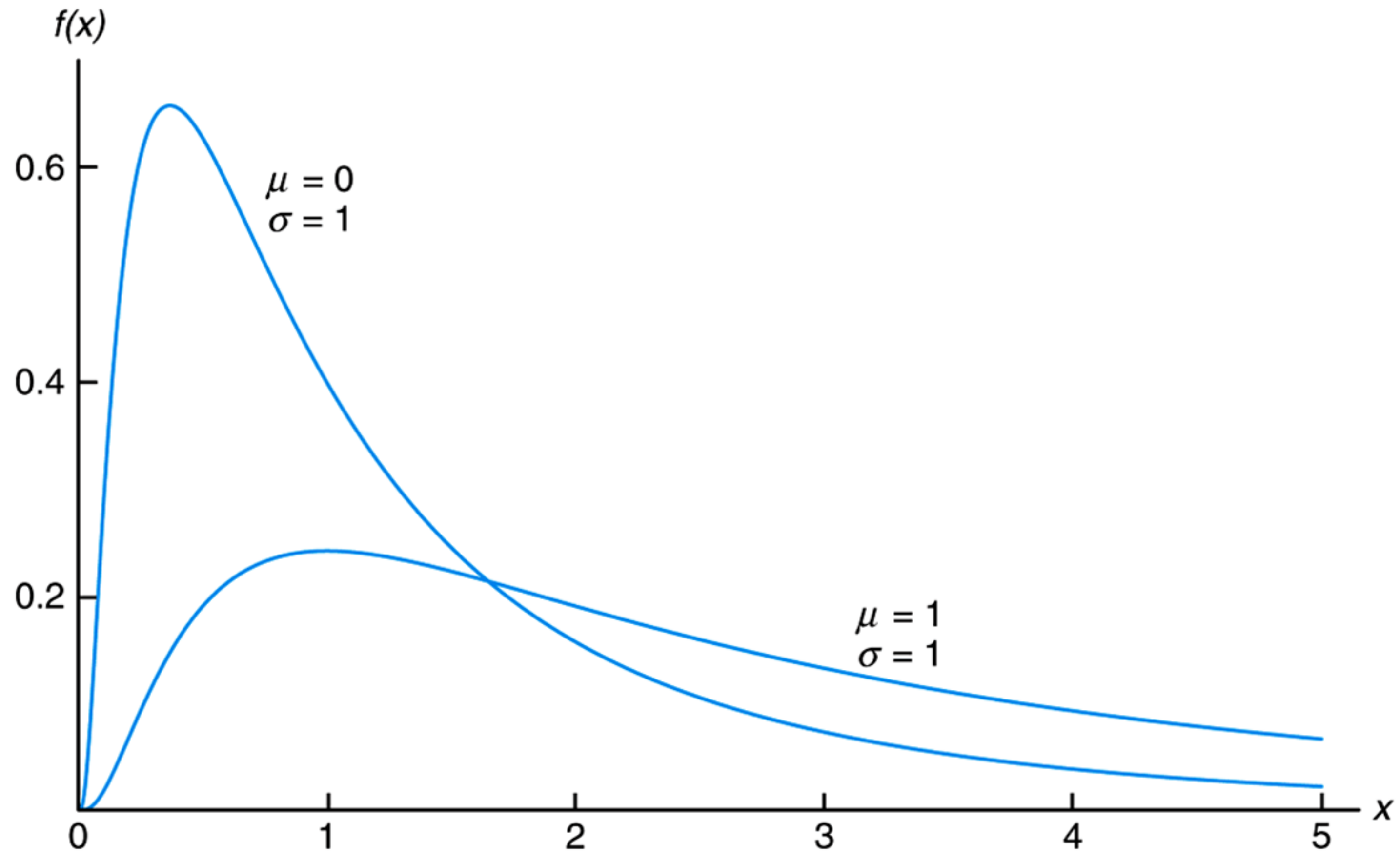
$$f(x;\mu,\sigma) = \begin{cases} \dfrac{1}{\sqrt{2\pi\sigma^2 x^2}} e^{-\frac{[\ln(x)-\mu]^2}{2\sigma^2}}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

where $Y = \ln(X)$.

The mean and variance of the log-normal distribution are

$$\mu = e^{\mu+\sigma^2/2} \text{ and } \sigma^2 = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$$

Log-Normal Distribution



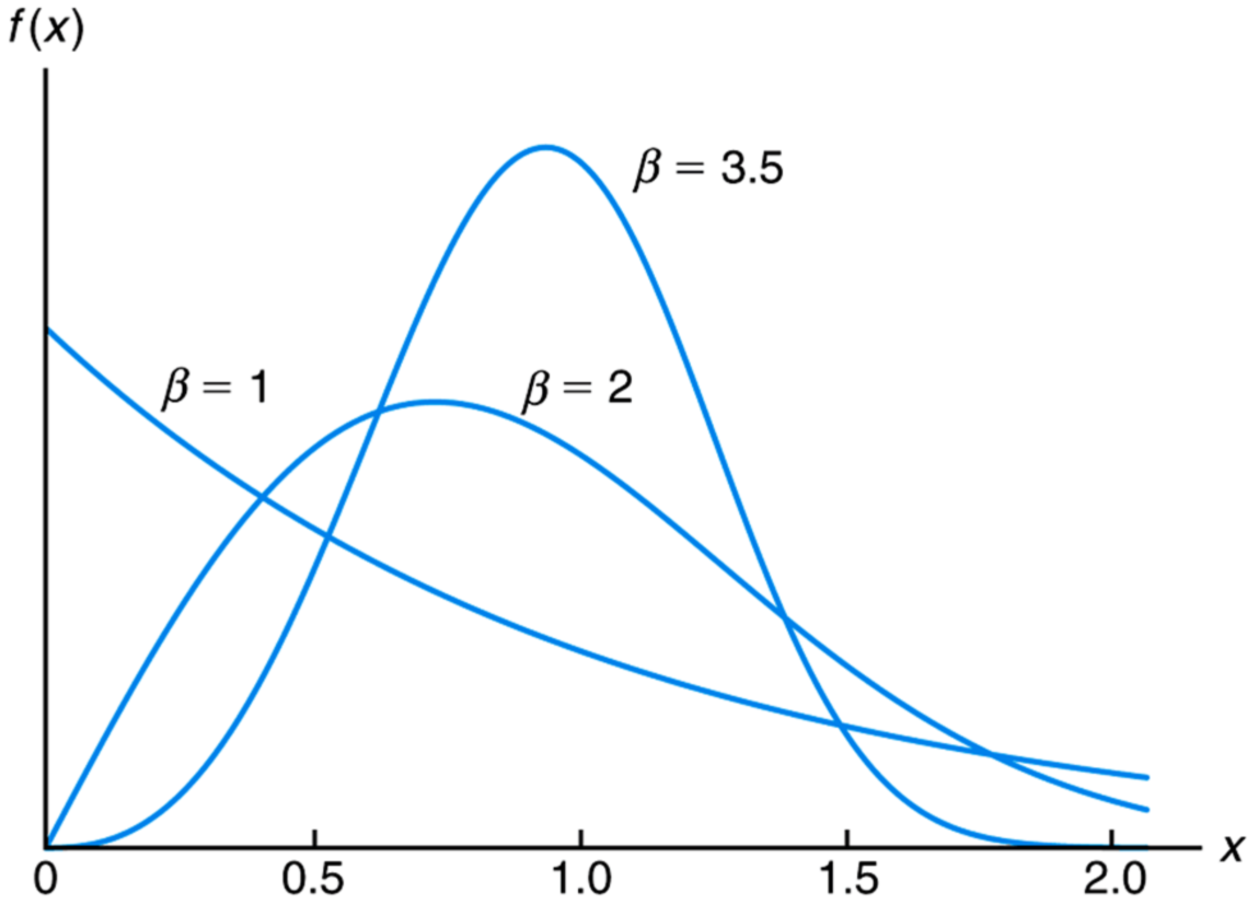
Weibull Distribution (α = 1)

PDF of the Weibull distribution is given by

$$f(x;\alpha,\beta) = \begin{cases} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Where $\alpha > 0$ and $\beta > 0$.

Weibull Distribution (α = 1)



Weibull Distribution

The mean and variance of the Weibull distribution are

$$\mu = \alpha^{-1/\beta}\,\Gamma\left(1 + \frac{1}{\beta}\right)$$

and

$$\sigma^2 = \alpha^{-2/\beta}\left\{\Gamma\left(1 + \frac{2}{\beta}\right) - \left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2\right\}$$

End of Lecture

Thank you! Questions?