Probability and Statistics
Lecture 5: Some Discrete Distributions

to accompany

Probability and Statistics for Engineers and Scientists

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Binomial and Multinomial Distributions

The probability distribution of the binomial RV X (defined as the # of successes in n independent trials each with success probability p) is given by,

$$b(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0,1, \dots, n$$

Binomial and Multinomial Distributions

Example 5.1: The probability that a certain component will survive a shock test is 3/4. Find the probability that exactly 2 of the next 4 components tested survive.

That is a binomial RV with parameters 4 and 3/4. We can then write

$$b\left(x=2;4,\frac{3}{4}\right)=\binom{4}{2}\left(\frac{3}{4}\right)^{2}\left(1-\frac{3}{4}\right)^{4-2}=\frac{27}{128}$$

Binomial and Multinomial Distributions

Example 5.3: A supplier indicates that its defective rate is 3%.

(a) The inspector randomly picks 20 items from a shipment. What is the probability that there will be at least 1 defective item among these 20?

(b) Suppose that the retailer receives 10 shipments in a month and the inspector randomly picks 20 items per shipment. What is the probability that there are 3 shipments each containing at least 1 defective device among 20 items?

Binomial and Multinomial Distributions

(a) We let X be the # of defectives among 20 items.

$$
\begin{aligned}
P\{X \geq 1\} &= 1 - P\{X = 0\} \\
&= 1 - b(x = 0; 20, 0.03) \\
&= 1 - \binom{20}{0}(0.03)^0(1 - 0.03)^{20-0} \\
&= 0.4562
\end{aligned}
$$

Binomial and Multinomial Distributions

(b) This time we can consider each shipment as a Bernoulli RV with $p = 0.4562$, and if we assume the shipments are independent, we can let Y be the # of shipments containing at least 1 defective item.

$$
\begin{aligned}
P\{Y = 3\} &= b(y = 3; 10{,}0.4562) \\
&= \binom{10}{3}(0.4562)^3(1 - 0.4562)^{10-3} \\
&= 0.1602
\end{aligned}
$$

Binomial and Multinomial Distributions

If a trial can result in the k outcomes $E_1, E_2, \dots, E_k$ with probabilities $p_1, p_2, \dots, p_k$, then, the probability distribution of the RVs $X_1, X_2, \dots, X_k$, representing the # of occurrences for the outcomes in n independent trials, is given by

$$f(x_1, x_2, \dots, x_k; p_1, p_2, \dots, p_k, n) = \binom{n}{x_1, x_2, \dots, x_k} p_1^{x_1} p_2^{x_2} \dots p_k^{x_k}$$

where

$$\sum_{i=1}^{k} x_i = n \text{ and } \sum_{i=1}^{k} p_i = 1$$

Binomial and Multinomial Distributions

Example 5.7: The complexity of arrivals and departures of planes at an airport is such that computer simulation is often used to model the ideal conditions. For a certain airport with 3 runways, it is known that in the ideal setting the following are the probabilities that the individual runways are accessed by a randomly arriving airplane:

Runway 1: $p_1 = 2/9$
Runway 2: $p_2 = 1/6$
Runway 3: $p_1 = 11/18$

Binomial and Multinomial Distributions

What is the probability that 6 randomly arriving airplanes are distributed in the following fashion:

Runway 1: 2 airplanes
Runway 2: 1 airplane
Runway 3: 3 airplanes

Binomial and Multinomial Distributions

Using the multinomial distribution, we have

$$f\left(x_1 = 2, x_2 = 1, x_3 = 3; \frac{2}{9}, \frac{1}{6}, \frac{11}{18}, 6\right) = \binom{6}{2,1,3}\left(\frac{2}{9}\right)^2\left(\frac{1}{6}\right)^1\left(\frac{11}{18}\right)^3$$
$$= \frac{6!}{2!\,1!\,3!}\left(\frac{2}{9}\right)^2\left(\frac{1}{6}\right)^1\left(\frac{11}{18}\right)^3$$
$$= 0.1127$$

Binomial and Multinomial Distributions

The mean and variance of the binomial distribution $b(x; n, p)$ are

$$\mu = np \text{ and } \sigma^2 = npq.$$

Hyper-Geometric Distribution

The mean and variance of the hypergeometric distribution $h(x; N, n, k)$ are

$$\mu = \frac{nk}{N} \text{ and } \sigma^2 = \frac{N-n}{N-1} \cdot n \cdot \frac{k}{N}\left(1 - \frac{k}{N}\right).$$

Hyper-Geometric Distribution

Example 5.9: Lots of 40 components are deemed unacceptable if they contain 3 or more defectives. The procedure for sampling a lot is to select 5 components at random and to reject the lot if 1 defective is found. What is the probability that exactly 1 defective is found in the sample if there are 3 defective in the lot?

We can write

$$f(x=1;40,5,3)=\frac{\binom{3}{1}\binom{37}{4}}{\binom{40}{5}}=0.3011$$

Multivariate Hyper-Geometric Distribution

Example 5.9: Lots of 40 components are deemed unacceptable if they contain 3 or more defectives. The procedure for sampling a lot is to select 5 components at random and to reject the lot if 1 defective is found. What is the probability that exactly 1 defective is found in the sample if there are 3 defectives in the lot?

We can write

$$f(x = 1; 40,5,3) = \frac{\binom{3}{1}\binom{37}{4}}{\binom{40}{5}} = 0.3011$$

Negative Binomial and Geometric Distribution

The probability distribution of negative binomial RV X, the number of trials until the kth success occurs, is

$$f(x; k, p) = \binom{x-1}{k-1} p^k (1-p)^{x-k}, \quad x = k+1, k+2, \ldots$$

Negative Binomial and Geometric Distribution

Example 5.14: In a championship series, the team that wins 4 games out of 7 is the winner. Assume that team A has probability of 0.55 winning a game over team B. What is the probability that team A will win the series in 6 matches?

We can write

$$f(x = 6; 4{,}0.55) = \binom{5}{3}(0.55)^4(1 - 0.55)^{6-4} = 0.1853$$

Negative Binomial and Geometric Distribution

The probability distribution of geometric RV X, the number of trials until the first success occurs, is

$$f(x;p) = p(1-p)^{x-1}, \quad x = 1,2,\ldots$$

Negative Binomial and Geometric Distribution

Example 5.16: At a busy time, a telephone exchange is very near capacity, so callers have difficulty placing their calls. It may be of interest to know the number of attempts necessary in order to make a connection during a busy time if we have a successful call probability of 0.05. In other words, find the probability that 5 attempts are necessary to for a successful call.

We have

$$f(x = 5, 0.05) = 0.05(1 - 0.05)^{5-1} = 0.041$$

Negative Binomial and Geometric Distribution

The mean and variance of a random variable following the geometric distribution are

$$\mu = \frac{1}{p} \text{ and } \sigma^2 = \frac{1-p}{p^2}.$$

Poisson Distribution

The probability distribution of a Poisson RV X, the # of outcomes occurring in a given time interval t when the average # of outcomes is λ per unit time, is

$$f(x;\lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}, \quad x = 0,1,\ldots$$

Poisson Distribution

Example 5.17: During a laboratory experiment, the average # of radioactive particles passing through a counter in 1 millisecond is 4. What is the probability that 6 particles enter the counter in a given millisecond?

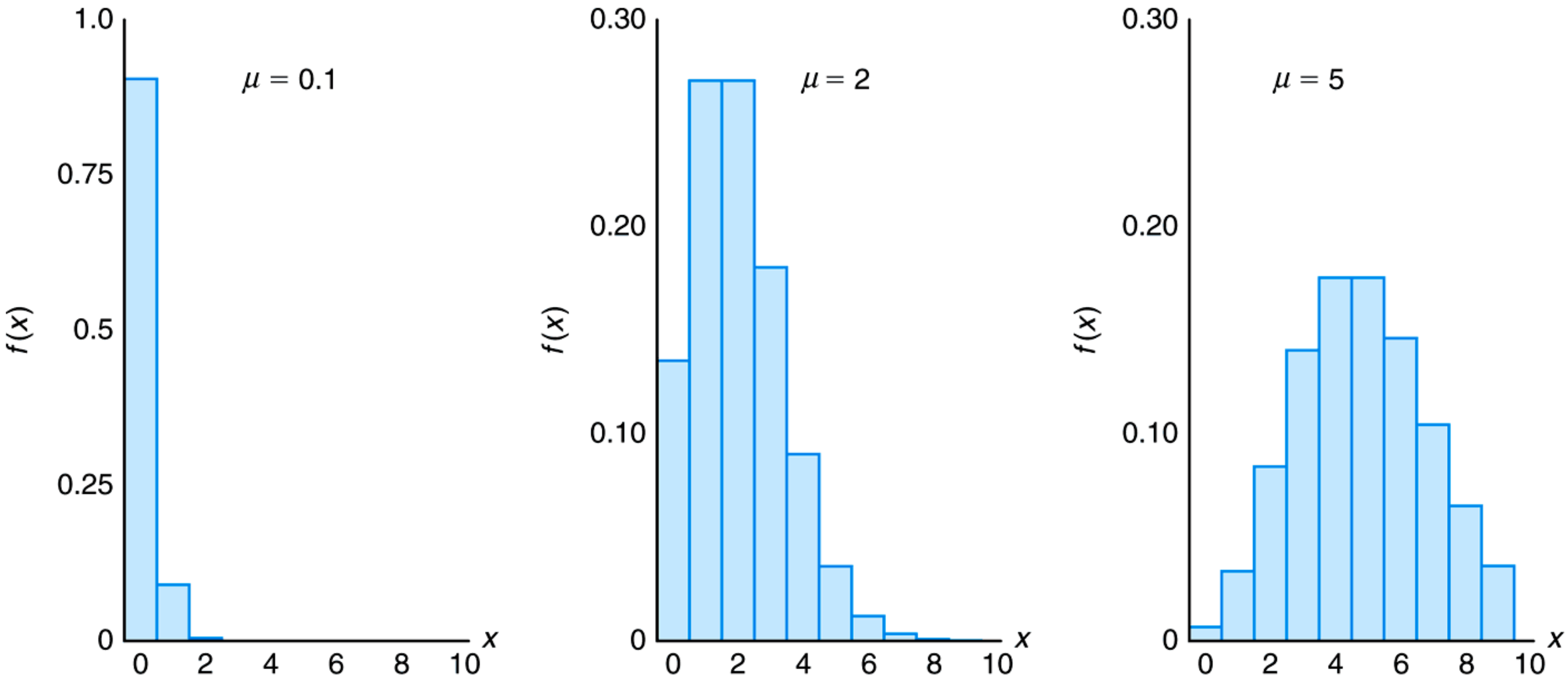
We have

$$f(x = 6,4) = \frac{e^{-4}4^{6}}{6!} = 0.1042$$

Poisson Distribution

$\boxed{\text{Both the mean and the variance of the Poisson distribution } p(x; \lambda t) \text{ are } \lambda t.}$

Poisson Distributions for Different Means



Poisson Approximation to Binomial

When $n \to \infty$ and $p \to 0$, we let $np = \mu$, and use the following approximation where f_1 and f_2 are binomial and Poisson probability distribution functions, respectively:

$$f_1(x; n, p) \to f_2(x; \mu)$$

Poisson Approximation to Binomial

Let X be a binomial random variable with probability distribution $b(x; n, p)$. When $n \to \infty$, $p \to 0$, and $np \overset{n\to\infty}{\longrightarrow} \mu$ remains constant,

$$b(x; n, p) \overset{n\to\infty}{\longrightarrow} p(x; \mu).$$

Poisson Approximation to Binomial

Example 5.19: In a certain facility, the probability of an accident on any given day is 0.005 and accidents are independent of each other. What is the probability that in any given period of 400 days there will be an accident on a day?

Poisson Approximation to Binomial

We let X as the # of accidents, a binomial RV with parameters 400 and 0.005. We can then let $\mu = (400)(0.005) = 2$ and write

$$
\begin{aligned}
P\{X = 1\} = f_1(x = 1{,}400{,}0.05) &= \binom{400}{1}(0.005)^1(0.095)^{399} \\
&\cong f_2(x = 1; 2) \\
&= \frac{e^{-2}2^1}{1!} = 0.271
\end{aligned}
$$

End of Lecture

Thank you! Questions?