Probability and Statistics
Lecture 4: Expectation

to accompany

Probability and Statistics for Engineers and Scientists

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Expectation of an RV

Let X be a random variable with probability distribution $f(x)$. The **mean**, or **expected value**, of X is

$$\mu = E(X) = \sum_x x f(x)$$

if X is discrete, and

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

if X is continuous.

Expectation of an RV

Example 4.1: A lot of 7 components of which 4 good and 3 bad is sampled by a quality inspector. A sample of 3 is taken by the inspector. What is the expected value of the # of good components in the sample?

$$f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}; \quad x = 0,1,2,3$$

$f(0) = 1/35$; $f(1) = 12/35$; $f(2) = 18/35$ and $f(3) = 4/35$.

$$\mu = E(X) = (0)\frac{1}{35} + (1)\frac{12}{35} + (2)\frac{18}{35} + (3)\frac{4}{35} = \frac{12}{7}$$

Expectation of an RV

Example 4.3:

$$f(x) = \begin{cases} \dfrac{20,000}{x^3}; & x > 100 \\ 0 & ow \end{cases}$$

$$\mu = E(X) = \int_{100}^{\infty} \frac{20,000dx}{x^2} = 200$$

Expectation of a Function of an RV

Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is

$$\mu_{g(X)} = E[g(X)] = \sum_x g(x)f(x)$$

if X is discrete, and

$$\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx$$

if X is continuous.

Expectation of Functions of RVs

Let X and Y be random variables with joint probability distribution $f(x, y)$. The mean, or expected value, of the random variable $g(X, Y)$ is

$$\mu_{g(X,Y)} = E[g(X, Y)] = \sum_{x} \sum_{y} g(x, y) f(x, y)$$

if X and Y are discrete, and

$$\mu_{g(X,Y)} = E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) \; dx \; dy$$

if X and Y are continuous.

Expectation of Functions of RVs (Example)

Example 4.6: 2 balls are selected at random from a box that contains 3 blue, 2 red and 3 green pens. Let X and Y be the # of blue and red pens selected.

$$f(x) = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{\binom{8}{2}}; \quad x = 0,1,2; \quad y = 0,1,2; \quad 0 \leq x+y \leq 2$$

Expectation of Functions of RVs (Example)

$f(x,y)$		$x=0$	$x=1$	$x=2$	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Expectation of Functions of RVs (Example)

$$E(XY) = \sum_{x=0}^{2} \sum_{y=0}^{2} xyf(x,y) = f(1,1) = \frac{3}{14}$$

Expectation of Functions of RVs (Example)

Find $E(Y/X)$ if

$$f(x,y) = \begin{cases} \dfrac{x(1+3y^2)}{4} & 0 < x < 2; 0 < y < 1 \\ 0 & ow \end{cases}$$

$$E\left(\frac{Y}{X}\right) = \int_0^1 \int_0^2 \frac{y(1+3y^2)}{4} dx\, dy = \int_0^1 \frac{y+3y^3}{2} dy = \frac{5}{8}$$

Variance of an RV

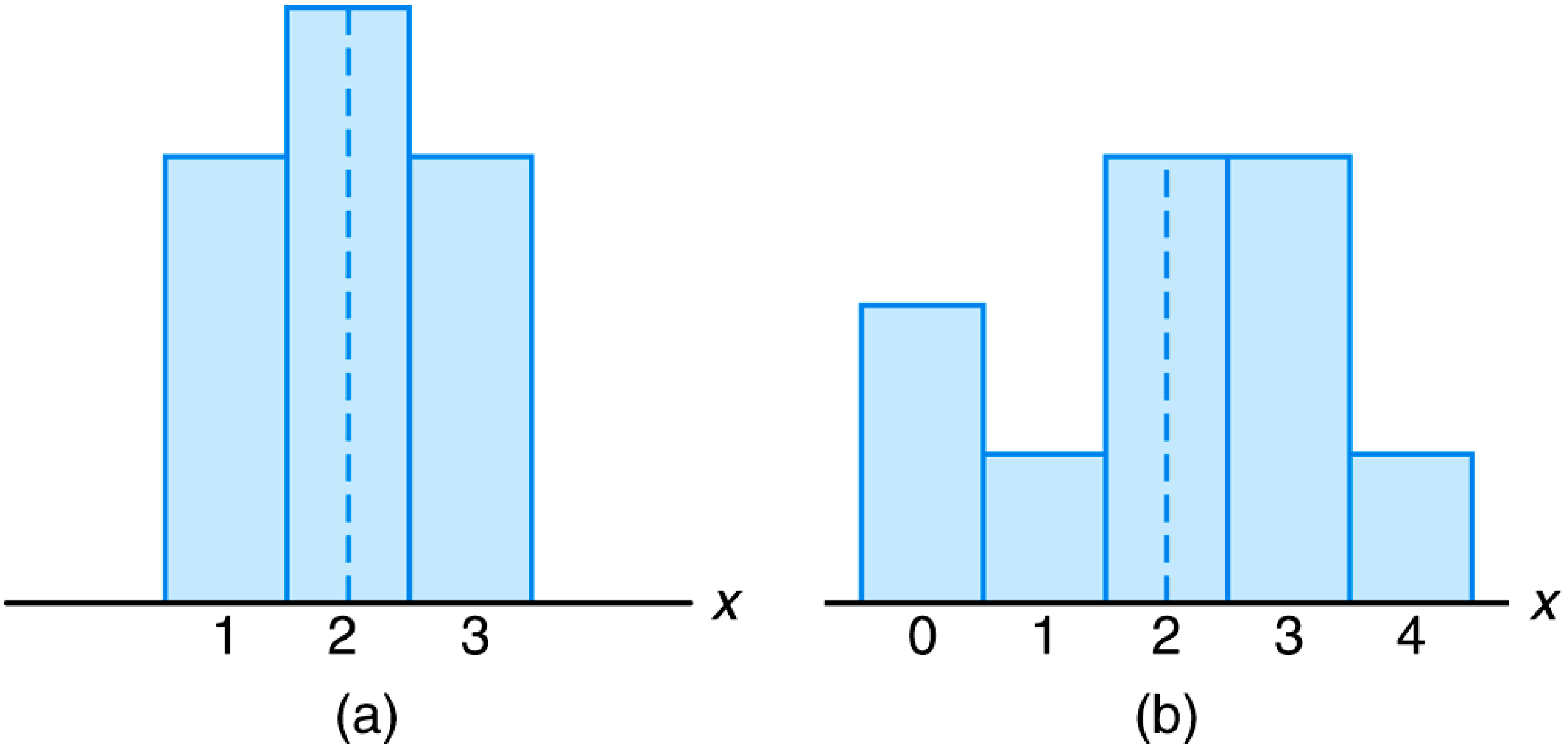
Let X be a random variable with probability distribution $f(x)$ and mean μ. The variance of X is

$$\sigma^2 = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x), \qquad \text{if } X \text{ is discrete, and}$$

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx, \qquad \text{if } X \text{ is continuous.}$$

The positive square root of the variance, σ, is called the **standard deviation** of X.

Variance of an RV (Example)



Variance of an RV

The variance of a random variable X is

$$\sigma^2 = E(X^2) - \mu^2.$$

Variance of an RV (Example)

x	0	1	2	3
$f(x)$	.51	.38	.10	.01

$$\mu = E(X) = 0(.51) + 1(.38) + 2(.10) + 3(.01) = 0.61$$

$$E(X^2) = 0(.51) + 1(.38) + 4(.10) + 9(.01) = 0.87$$

$$\sigma^2 = Var(X) = E(X^2) - [E(X)]^2 = 0.87 - (0.61)^2 = 0.4979$$

Variance of an RV (Example)

Find the variance of X if

$$f(x) = \begin{cases} 2(x-1); & 1 < x < 2 \\ 0 & ow \end{cases}$$

$$\mu = E(X) = \int_{1}^{2} 2x(x-1)dx = \frac{5}{3}$$

$$E(X^2) = \int_{1}^{2} 2x^2(x-1)dx = \frac{17}{6}$$

$$\sigma^2 = Var(X) = E(X^2) - [E(X)]^2 = \frac{17}{6} - \left(\frac{5}{3}\right)^2 = \frac{1}{18}$$

Variance of a Function of an RV

Let X be a random variable with probability distribution $f(x)$. The variance of the random variable $g(X)$ is

$$\sigma^2_{g(X)} = E\{[g(X) - \mu_{g(X)}]^2\} = \sum_x [g(x) - \mu_{g(X)}]^2 f(x)$$

if X is discrete, and

$$\sigma^2_{g(X)} = E\{[g(X) - \mu_{g(X)}]^2\} = \int_{-\infty}^{\infty} [g(x) - \mu_{g(X)}]^2 f(x)\, dx$$

if X is continuous.

Variance of Functions of RVs

Let X and Y be random variables with joint probability distribution $f(x, y)$. The covariance of X and Y is

$$\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \sum_x \sum_y (x - \mu_X)(y - \mu_y) f(x, y)$$

if X and Y are discrete, and

$$\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_y) f(x, y)\, dx\, dy$$

if X and Y are continuous.

Covariance and Correlation Coefficient

The covariance of two random variables X and Y with means μ_X and μ_Y, respectively, is given by

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y.$$

Let X and Y be random variables with covariance σ_{XY} and standard deviations σ_X and σ_Y, respectively. The correlation coefficient of X and Y is

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}.$$

Means and Variances of Linear Combinations of Random Variables

If a and b are constants, then

$$E(aX + b) = aE(X) + b.$$

Setting $a = 0$, we see that $E(b) = b$.

Setting $b = 0$, we see that $E(aX) = aE(X)$.

Means and Variances of Linear Combinations of Random Variables

The expected value of the sum or difference of two or more functions of a random variable X is the sum or difference of the expected values of the functions. That is,

$$E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)].$$

The expected value of the sum or difference of two or more functions of the random variables X and Y is the sum or difference of the expected values of the functions. That is,

$$E[g(X,Y) \pm h(X,Y)] = E[g(X,Y)] \pm E[h(X,Y)].$$

Setting $g(X,Y) = g(X)$ and $h(X,Y) = h(Y)$, we see that

$$E[g(X) \pm h(Y)] = E[g(X)] \pm E[h(Y)].$$

Setting $g(X,Y) = X$ and $h(X,Y) = Y$, we see that

$$E[X \pm Y] = E[X] \pm E[Y].$$

Means and Variances of Linear Combinations of Random Variables

Let X and Y be two independent random variables. Then

$$E(XY) = E(X)E(Y).$$

Let X and Y be two independent random variables. Then $\sigma_{XY} = 0$.

If X and Y are random variables with joint probability distribution $f(x, y)$ and a, b, and c are constants, then

$$\sigma^2_{aX+bY+c} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma_{XY}.$$

Setting $b = 0$, we see that

$$\sigma^2_{aX+c} = a^2\sigma^2_X = a^2\sigma^2.$$

Setting $b = 0$ and $c = 0$, we see that

$$\sigma^2_{aX} = a^2\sigma^2_X = a^2\sigma^2.$$

Means and Variances of Linear Combinations of Random Variables

If X and Y are independent random variables, then

$$\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y.$$

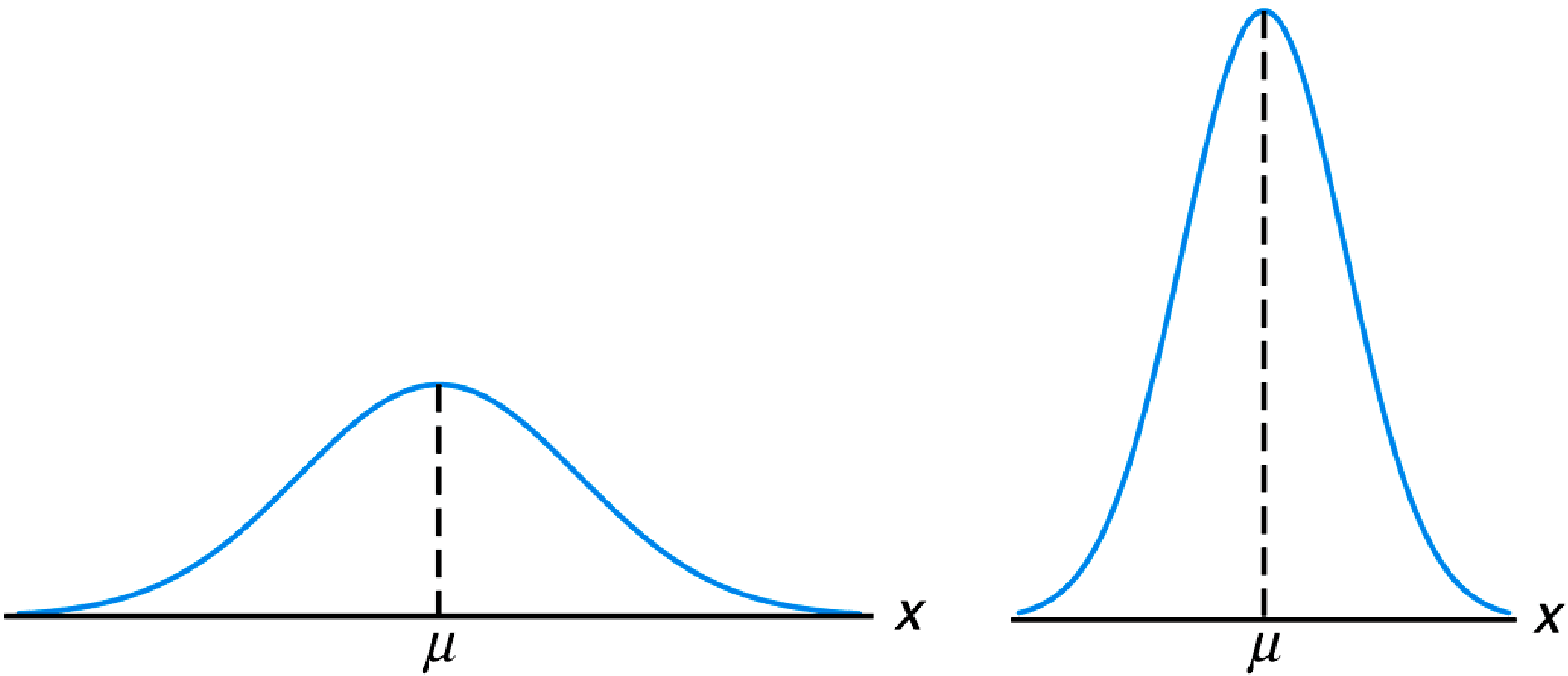
If X and Y are independent random variables, then

$$\sigma^2_{aX-bY} = a^2\sigma^2_X + b^2\sigma^2_Y.$$

If $X_1, X_2, \ldots, X_n$ are independent random variables, then

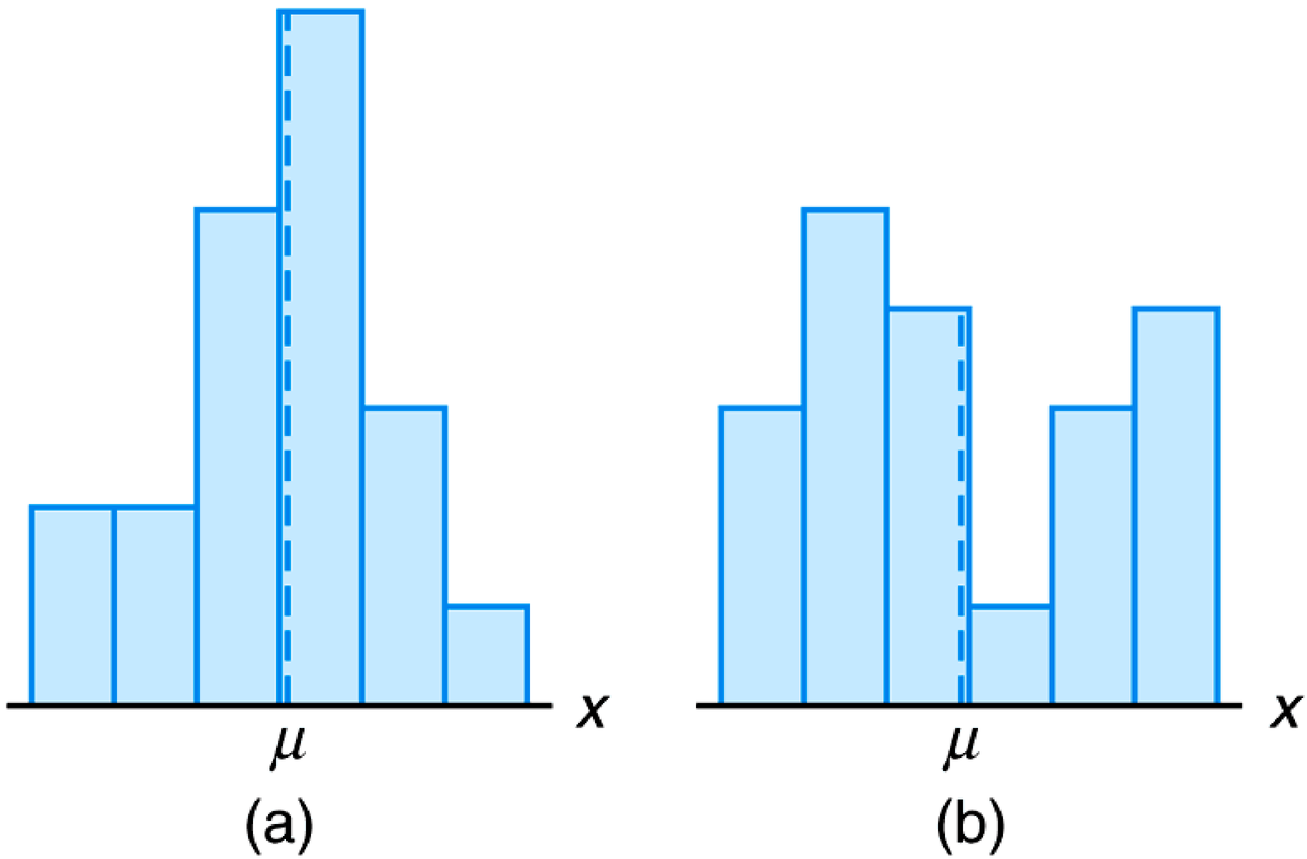
$$\sigma^2_{a_1X_1+a_2X_2+\cdots+a_nX_n} = a_1^2\sigma^2_{X_1} + a_2^2\sigma^2_{X_2} + \cdots + a_n^2\sigma^2_{X_n}.$$

Chebyshev's Theorem



(a) (b)

Chebyshev's Theorem



Chebyshev's Theorem

(Chebyshev's Theorem) The probability that any random variable X will assume a value within k standard deviations of the mean is at least $1 - 1/k^2$. That is,

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}.$$

Chebyshev's Theorem (Example)

An RV X has a mean $\mu = 8$ and a variance $\sigma^2 = 9$ with an unknown distribution. Using Chebyshev's theorem, we can find, for instance,

$$P\{-4 < X < 20\} = P\{8 - (4)(3) < X < 8 + (4)(3)\} \geq 1 - \frac{1}{16} = \frac{15}{16}$$

End of Lecture

Thank you! Questions?