Probability and Statistics
Lecture 2: Probability

to accompany

Probability and Statistics for Engineers and Scientists

Fatih Cavdur

Chapter 2: Probability

- Sample Space
- Events
- Counting Sample Points
- Probability of an Event
- Conditional Probability
- Bayes' Rule

Sample Space

The set of all possible outcomes of a statistical experiment is called the **sample space** and is represented by the symbol S.

Sample Space

- When we flip a coin,

$$S = \{h, t\}$$

- When we toss a die,

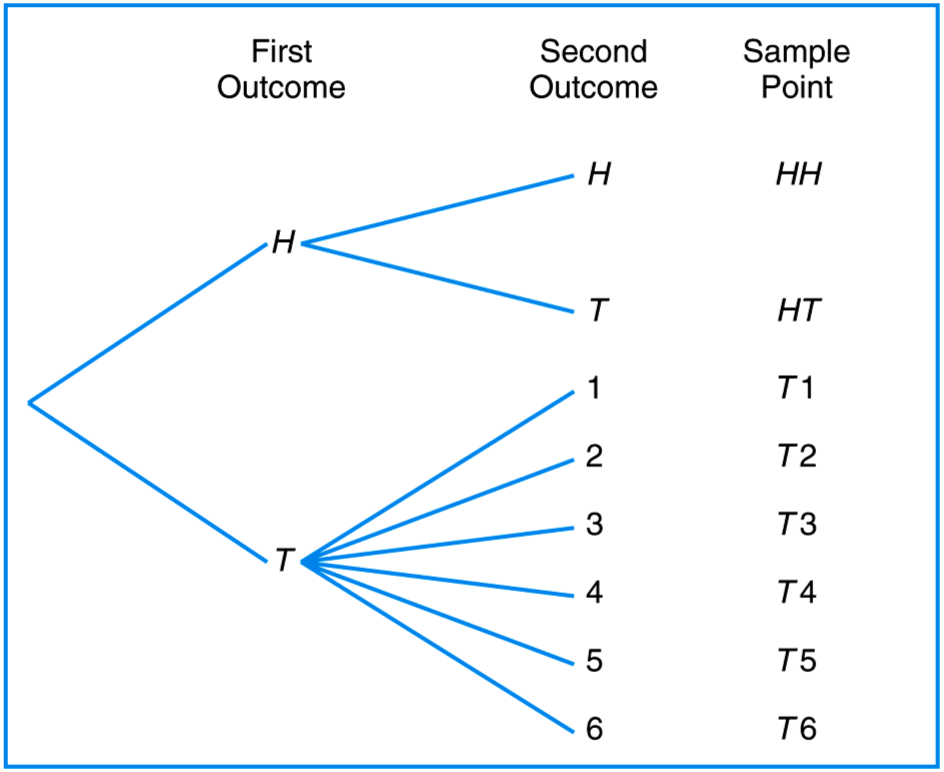
$$S = \{1,2, \dots, 6\}$$

- If the sample space is the interior of a circle of radius 3 with the center at the origin

$$S = \{(x, y) | x^2 + y^2 \leq 9\}$$

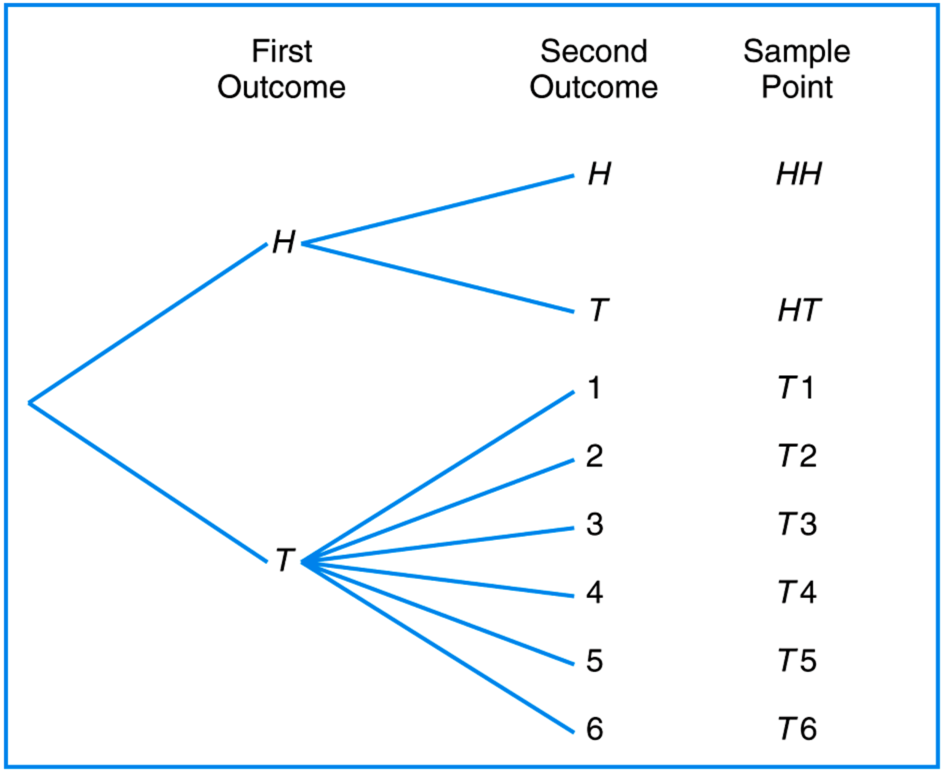
Sample Space

- An experiment is performed as follows: We first flip a coin. If the coin is head, we flip it again. Otherwise, we toss a die. An experiment like this can be represented by the following tree diagram.



Sample Space

- Suppose that 3 items are selected at random from a manufacturing process. Each one is inspected and classified as defective (D) or non-defective (N). The corresponding tree diagram for the process is shown.



Events

An **event** is a subset of a sample space.

The **complement** of an event A with respect to S is the subset of all elements of S that are not in A. We denote the complement of A by the symbol A'.

The **intersection** of two events A and B, denoted by the symbol $A \cap B$, is the event containing all elements that are common to A and B.

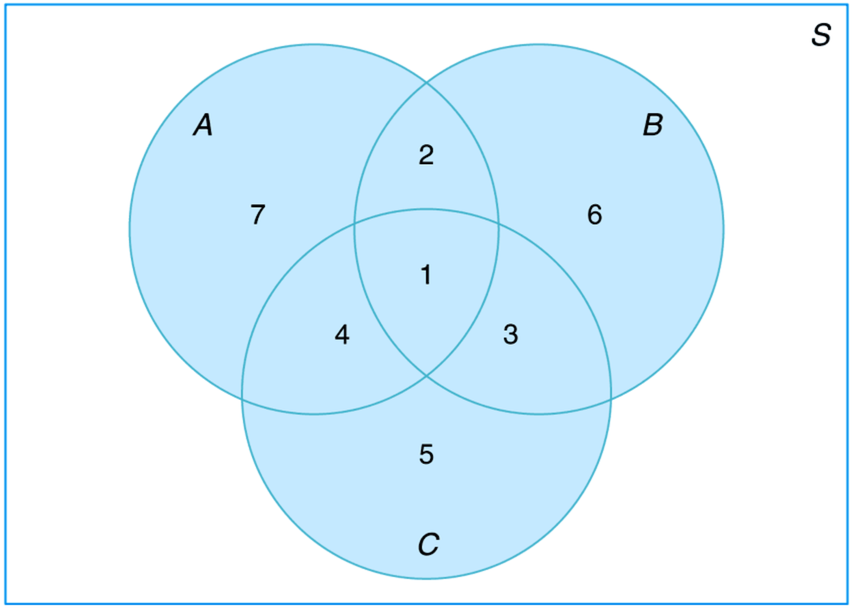
Two events A and B are **mutually exclusive**, or **disjoint**, if $A \cap B = \phi$, that is, if A and B have no elements in common.

The **union** of the two events A and B, denoted by the symbol $A \cup B$, is the event containing all the elements that belong to A or B or both.

Events Represented by Venn Diagrams

Let

$$A = \{1,2,4,7\}$$
$$B = \{1,2,3,6\}$$
$$C = \{1,3,4,5\}$$



We can then write

$$A \cup B = \{1,2,3,4,6,7\}$$
$$A \cup C = \{1,2,3,4,5,7\}$$
$$A \cap B = \{1,2\}$$
$$A \cap B' = \{4,7\}$$
$$(A \cup B) \cap C' = \{2,6,7\}$$

Counting Sample Points

If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \cdots n_k$ ways.

A **permutation** is an arrangement of all or part of a set of objects.

For any non-negative integer n, $n!$, called "n factorial," is defined as

$$n! = n(n-1)\cdots(2)(1),$$

with special case $0! = 1$.

Counting Sample Points

The number of permutations of n objects is $n!$.

The number of permutations of n distinct objects taken r at a time is

$$_nP_r = \frac{n!}{(n-r)!}.$$

The number of permutations of n objects arranged in a circle is $(n-1)!$.

The number of distinct permutations of n things of which n_1 are of one kind, n_2 of a second kind, $\ldots$, n_k of a kth kind is

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Counting Sample Points

The number of distinct permutations of n things of which n_1 are of one kind, n_2 of a second kind, ..., n_k of a kth kind is

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

The number of ways of partitioning a set of n objects into r cells with n_1 elements in the first cell, n_2 elements in the second, and so forth, is

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!n_2!\cdots n_r!},$$

where $n_1 + n_2 + \cdots + n_r = n$.

The number of combinations of n distinct objects taken r at a time is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Probability of an Event

The **probability** of an event A is the sum of the weights of all sample points in A. Therefore,

$$0 \leq P(A) \leq 1, \quad P(\phi) = 0, \quad \text{and} \quad P(S) = 1.$$

Furthermore, if $A_1, A_2, A_3, \ldots$ is a sequence of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots .$$

If an experiment can result in any one of N different equally likely outcomes, and if exactly n of these outcomes correspond to event A, then the probability of event A is

$$P(A) = \frac{n}{N}.$$

Probability of Union of Events

If A and B are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

If $A_1,\ A_2, \ldots, A_n$ are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

Probability of Union of Events

If $A_1,\ A_2,\ldots,A_n$ is a partition of sample space S, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) = P(S) = 1.$$

For three events A, B, and C,

$$\begin{aligned} P(A \cup B \cup C) = {} & P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C). \end{aligned}$$

If A and A' are complementary events, then

$$P(A) + P(A') = 1.$$

Conditional Probability

The conditional probability of B, given A, denoted by $P(B|A)$, is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided} \quad P(A) > 0.$$

Conditional Probability

- Suppose that our sample space is the population of adults in a small town who have completed the requirements for a collage degree. If we categorize them according to gender and employment status, we have the following results:

	Employed	**Unemployed**	**Total**
Male	460	40	500
Female	140	260	400
Total	600	300	900

Conditional Probability

- If we select someone at random, and define the following events:
  - M: a man is chosen
  - W: a woman is chosen
  - E: the one chosen is employed
  - U: the one chosen is unemployed
- We can then, for instance, compute the probability that a man is chosen given that the one chosen is employed as follows:

$$P(M|E) = \frac{P(M \cap E)}{P(E)} = \frac{460}{600} = \frac{23}{30}$$

Independent Events

Two events A and B are **independent** if and only if

$$P(B|A) = P(B) \quad \text{or} \quad P(A|B) = P(A),$$

assuming the existences of the conditional probabilities. Otherwise, A and B are **dependent**.

If in an experiment the events A and B can both occur, then

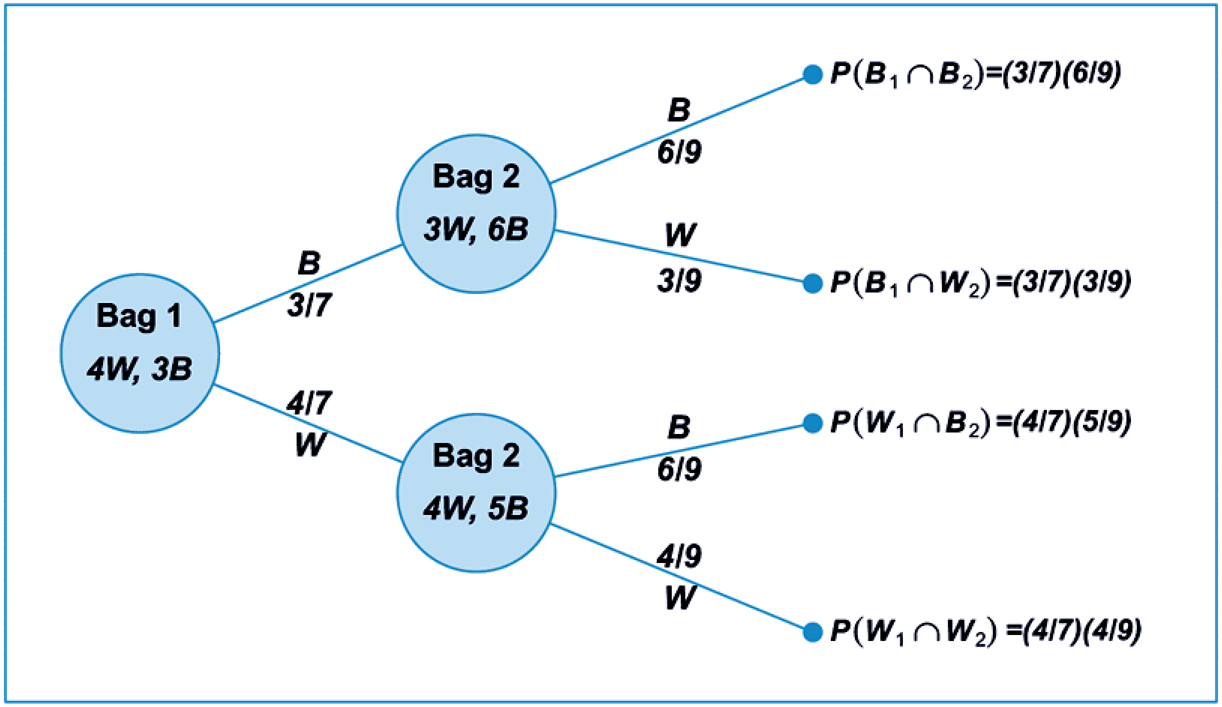
$$P(A \cap B) = P(A)P(B|A), \text{ provided } P(A) > 0.$$

Example 2.37

- A bag contains 4 white and 3 black balls, and another one contains 3 white and 5 black balls. A ball is drawn from the first bag and put unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?
- We first let B_1, B_2, W_1, W_2 be the events that the ball drawn from the first bag is black, from the second bag is black, etc.
- We then need to compute the probability of
$$P[(B_1 \cap B_2) \cup (W_1 \cap B_2)] = P(B_1 \cap B_2) + P(W_1 \cap B_2)$$
- Can we write the above expression? Why?

Example 2.37

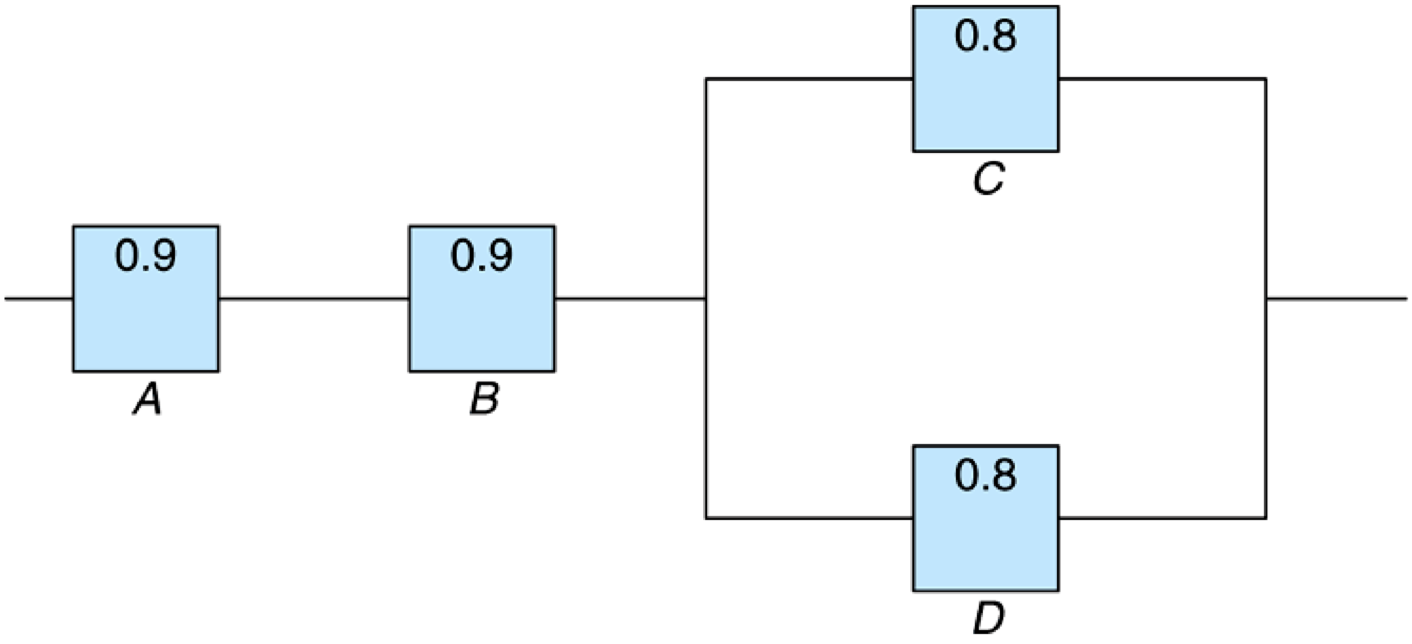
$$P(B_1 \cap B_2) + P(W_1 \cap B_2) = P(B_1)P(B_2|B_1) + P(B_1)P(B_2|B_1)$$
$$= \frac{3}{7}\frac{6}{9} + \frac{4}{7}\frac{5}{9}$$



Fatih Cavdur – fatihcavdur@uludag.edu.tr

Example 2.39

- Find the probability that
  - the following electrical system works.
  - component C does not work given that the system works.



Example 2.39

The desired probability is computed as follows:

$$
\begin{aligned}
P[A \cap B \cap (C \cup D)] &= P(A)P(B)P(C \cup D) \\
&= P(A)P(B)[1 - P(C')P(D')] \\
&= (.9)(.9)[1 - (1 - .8)(1 - .8)] \\
&= .7776
\end{aligned}
$$

Independent Events

If, in an experiment, the events $A_1, A_2, \ldots, A_k$ can occur, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k)$$
$$= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_k|A_1 \cap A_2 \cap \cdots \cap A_{k-1}).$$

If the events $A_1, A_2, \ldots, A_k$ are independent, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2) \cdots P(A_k).$$

A collection of events $\mathcal{A} = \{A_1, \ldots, A_n\}$ are mutually independent if for any subset of $\mathcal{A}$, $A_{i_1}, \ldots, A_{i_k}$, for $k \leq n$, we have

$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \cdots P(A_{i_k}).$$

Bayes' Rule

If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A of S,

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A|B_i).$$

(Bayes' Rule) If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A in S such that $P(A) \neq 0$,

$$P(B_r|A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)P(A|B_r)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)} \quad \text{for } r = 1, 2, \ldots, k.$$

Bayes' Rule

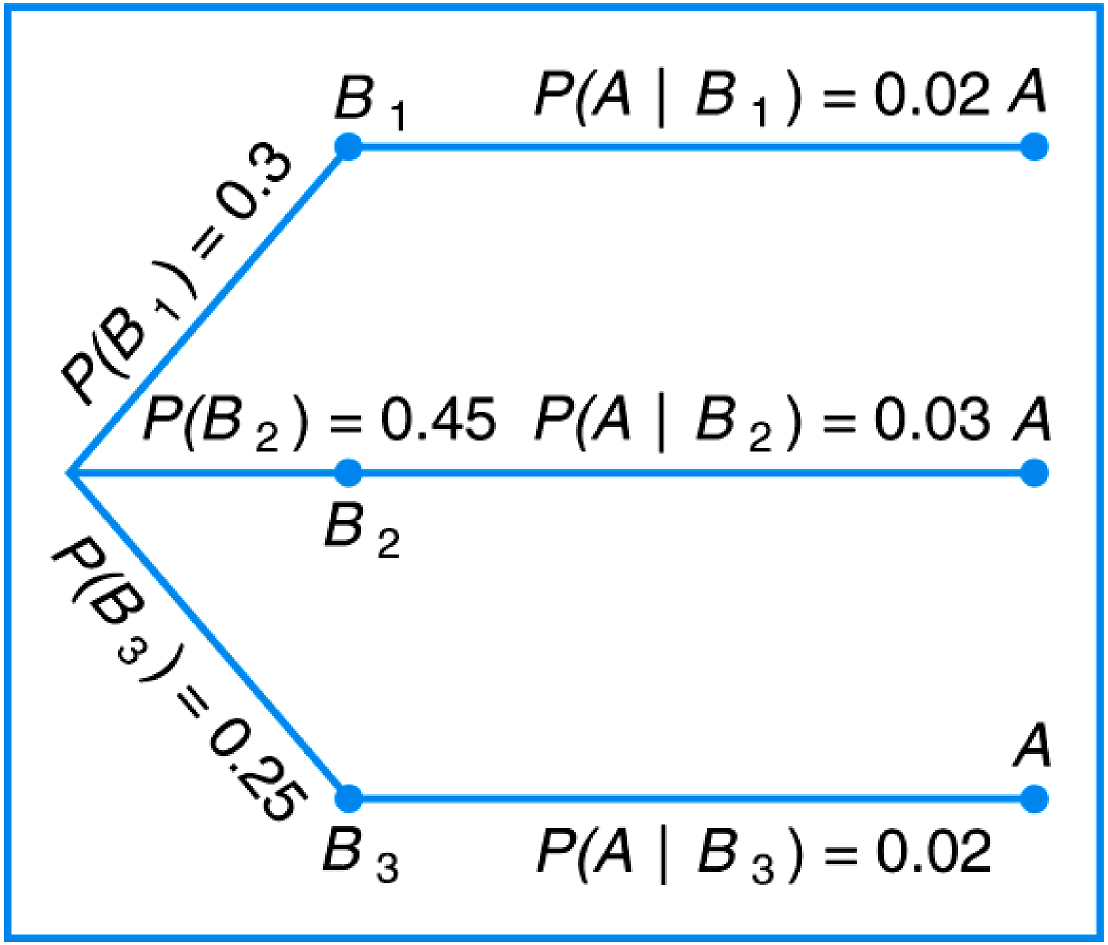
- In a plant machines M1, M2 and M3 make the 30%, 45% and 25% of the products, respectively. It is known that the defective rates of these machines are 2%, 3% and 2%, respectively. What is the probability that, if a randomly chosen product is defective, it was made by machine M3?

Bayes' Rule

- We need to compute the following probability using the Bayes' rule:

$$P(B_3|D) = \frac{P(B_3 \cap D)}{P(D)}$$
$$= \frac{P(B_3)P(D|B_3)}{P(B_3)P(D|B_3) + P(B_3)P(D|B_3) + P(B_3)P(D|B_3)}$$
$$= \frac{.005}{.006 + .0135 + .005}$$
$$= \frac{10}{49}$$

Bayes' Rule



B_1
$P(A \mid B_1) = 0.02$ A
$P(B_1) = 0.3$
$P(B_2) = 0.45$
$P(A \mid B_2) = 0.03$ A
B_2
$P(B_3) = 0.25$
A
B_3
$P(A \mid B_3) = 0.02$

End of Lecture

Thank you! Questions?