Renewal Theory

Stochastic Processes - Lecture Notes

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to accompany
Introduction to Probability Models
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Outline

Introduction

- Poisson Process is a special counting process for which the time between successive events are IID exponential RVs.
- If we generalize the distribution assumption so that the time between successive events are IID RVs with a general distribution), such a counting process is called a *renewal process*.

Introduction

Let $\{N(t), t \geq 0\}$ be a counting process and let X_n be the time between the $(n-1)$th and nth event of the process, $n \geq 1$.

Definition

If the sequence of non-negative RVs $\{X_1, X_2, \ldots\}$ is IID, then, the counting process $\{N(t), t \geq 0\}$ is said to be a renewal process.

Introduction

Example

For an example of renewal process, suppose that we have an infinite supply of light-bulbs with IID life times. Suppose also that we use a single light-bulb at a time, and when it fails, we immediately replace it with a new one. Under these conditions, $\{N(t), t \geq 0\}$ is a renewal process where $N(t)$ represents the number of light-bulbs that have failed by time t.

Introduction

For a renewal process with inter-arrival times $X_1, X_2, \ldots$, let

$$S_0 = 0, \text{ and } S_n = \sum_{i=1}^{n} X_i, n \geq 1$$

Let F be the inter-arrival distribution and assume that $F(0) = P_{X_n=0} < 1$. Let also $\mu = E(X_n)$, $n \geq 1$ be the mean time between successive arrivals. We want to show that an infinite number of arrivals cannot occur in a finite amount of time.

Introduction

To show this, first note that

$$N(t) = \max\{n : S_n \leq t\}$$

By the strong law of large numbers, it follows with probability 1, that

$$\lim_{n\to\infty} \frac{S_n}{n} \to \mu$$

However, since $\mu > 0$, $S_n \to \infty$ as $n \to \infty$, and thus, $S_n \leq t$ for at most a finite number of values of n, and hence, $N(t)$ must be finite.

Introduction

Although $N(t) < \infty$, $\forall t$, with probability 1, we can write

$$N(\infty) = \lim_{t\to\infty} N(t) = \infty$$

and, thus,

$$\begin{aligned} P\{N(\infty) < \infty\} &= P\{X_n = \infty, \exists n\} \\ &= P\left\{\bigcup_{n=1}^{\infty} X_n = \infty\right\} \leq \sum_{n=1}^{\infty} P\{X_n = \infty\} = 0 \end{aligned}$$

Distribution of the Number of Renewals

We note that the number of renewals by time t is greater than or equal to n if and only if the nth renewal occurs before time t. That is,

$$N(t) \geq n \Leftrightarrow S_n \leq t$$

and thus,

$$\begin{aligned} P\{N(t) = n\} &= P\{N(t) \geq n\} - P\{N(t) \geq n+1\} \\ &= P\{S_n \leq t\} - P\{S_{n+1} \leq t\} \end{aligned}$$

Distribution of the Number of Renewals

Since X_i, $i \geq 1$, are IID RVs with a common distribution F, it follows that, S_n is distributed as F_n, the n-fold convolution of F with itself. That is,

$$X_i \sim F(\cdot) \Rightarrow S_n = \sum_{i=1}^{n} X_i \sim F_n(\cdot)$$

and thus,

$$P\{N(t) = n\} = F_n(t) - F_{n+1}(t)$$

Distribution of the Number of Renewals

Example

Suppose that the time between arrivals is geometric. That is,

$$P\{X_n = i\} = p(1-p)^{i-1},\ i \geq 1$$

- $S_1 = X_1$ may be interpreted as the number of trials necessary to get the first success if each trial is independent and has a probability p of being a success.
- Similarly, S_n may be interpreted as the number of trials necessary to obtain n successes, and hence, has the negative binomial distribution.

$$P\{S_n = k\} = \begin{cases} \binom{k-1}{n-1} p^n (1-p)^{k-n}, & k \geq n \\ 0, & k < n \end{cases}$$

Distribution of the Number of Renewals

Example

We then have

$$\begin{aligned} P\{N_t = n\} = & \sum_{k=n}^{[t]} \binom{k-1}{n-1} p^n (1-p)^{k-n} \\ & - \sum_{k=n+1}^{[t]} \binom{k-1}{n} p^{n+1} (1-p)^{k-n-1} \end{aligned}$$

Equivalently, since an event independently occurs with probability p at each $t = 1, 2, \ldots,$

$$P\{N_t = n\} = \binom{[t]}{n} p^n (1-p)^{[t]-n}$$

Mean-Value or Renewal Function

We can calculate the mean value of $N(t)$ as

$$\begin{aligned} m(t) &= E[N(t)] \\ &= \sum_{n=1}^{\infty} P\{N(t) \geq n\} \\ &= \sum_{n=1}^{\infty} P\{S_n \leq t\} \\ &= \sum_{n=1}^{\infty} F_n(t) \end{aligned}$$

Mean-Value or Renewal Function

where we have used the fact that when X is non-negative and integer-valued, we have

$$\begin{aligned} E(X) &= \sum_{k=1}^{\infty} kP\{X = k\} \\ &= \sum_{k=1}^{\infty} \sum_{n=1}^{k} P\{X = k\} \\ &= \sum_{k=1}^{\infty} \sum_{k=n}^{\infty} P\{X = k\} \\ &= \sum_{n=1}^{\infty} P\{X \geq n\} \end{aligned}$$

Mean-Value or Renewal Function

- The function $m(t)$ is the *mean-value* or the *renewal* function.
- We can show that $m(t)$ uniquely determines the renewal process.
- Specifically, there is a one-to-one correspondence between the time-between-arrivals distribution F and the mean-value function $m(t)$.
- An integral equation, called as the renewal equation, satisfied by the renewal function can be obtained by conditioning on the time of the first renewal.
- The renewal equation can sometimes be solved to obtain the renewal function.

Renewal Equation

$$m(t) = E[N(t)] = \int_0^\infty E[N(t)|X_1 = x]f(x)dx$$

Since

$$E[N(t)|X_1 = x] = \begin{cases} 1 + E[N(t-x)], & \text{if } x < t \\ 0, & \text{if } x > t \end{cases}$$

$$m(t) = \int_0^t [1 + m(t-x)]f(x)dx = F(t) + \int_0^t m(t-x)f(x)dx$$

Renewal Equation

Example

The renewal equation can be solved when the time-between-arrivals is uniform. Assume that it is uniform between 0 and 1. When $t < 1$,

$$
\begin{aligned}
m(t) &= t + \int_0^t m(t-x)dx \quad (\text{let } y = t - x) \\
&= t + \int_0^t m(y)dy
\end{aligned}
$$

Renewal Equation

Example

By differentiating,

$$m(t) = t + \int_0^t m(y)dy \Rightarrow m'(t) = 1 + m(t)$$

and letting $h'(t) = 1 + m(t)$,

$$
\begin{aligned}
h'(t) = h(t) &\Rightarrow \log h(t) = t + c \\
& \qquad\qquad h(t) = ke^t \Rightarrow m(t) = ke^t + 1
\end{aligned}
$$

Since $m(0) = 0$, we see that $k = 1$, and hence,

$$m(t) = e^t - 1, \quad 0 \leq t \leq 1$$

Limit Theorems and Applications

We know that $N(t)$ goes to infinity as t goes to infinity. That is,

$$\lim_{t\to\infty} N(t) = \infty$$

and want to know about the rate at which $N(t)$ goes to infinity as the following states:

Proposition

The rate at which $N(t)$ goes to infinity, with probability 1, is

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}$$

where μ is the mean time between arrivals.

Limit Theorems and Applications

Since $S_{N(t)}$ is the time of the last renewal prior to or at time t, and $S_{N(t)+1}$ is the time of the first renewal after time t, we have

$$S_{N(t)} \leq t < S_{N(t)+1}$$

or

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}$$

Since $S_{N(t)}/N(t) \to \mu$ as $N(t) \to \infty$, but since $N(t) \to \infty$ when $t \to \infty$, it follows by the Strong Law of Large Numbers that

$$\lim_{t\to\infty} \frac{S_{N(t)}}{N(t)} \to \mu$$

Limit Theorems and Applications

Furthermore, by writing,

$$\frac{S_{N(t)}+1}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1}\frac{N(t)+1}{N(t)}$$

we have that

$$\lim_{t\to\infty} \frac{S_{N(t)+1}}{N(t)+1} \to \mu$$

by the same reasoning and

$$\lim_{t\to\infty} \frac{N(t)+1}{N(t)} \to 1 \Rightarrow \lim_{t\to\infty} \frac{S_{N(t)+1}}{N(t)} \to \mu$$

Limit Theorems and Applications

We note that,

(i) Preceding propositions are true when μ is infinite.

(ii) The number $1/\mu$ is called as the rate of renewal process.

(iii) The average rate at which the renewals occur is 1 per every μ time units.

Limit Theorems and Applications

Elementary Renewal Theorem

$$\lim_{t\to\infty} \frac{m(t)}{t} \to \frac{1}{\mu}$$

Limit Theorems and Applications

Example

Let U be a uniform RV between 0 and 1, and define the RV Y_n, $n \geq 1$, by

$$Y_n = \begin{cases} 0, & \text{if } U > 1/n \\ n, & \text{if } U \leq 1/n \end{cases} \Rightarrow \lim_{n\to\infty} Y_n = 0$$

However,

$$E[Y(n)] = nP\left\{U \leq \frac{1}{n}\right\} = n\frac{1}{n} = 1$$

Hence, even though the sequence of RVs Y_n converges to 0, their expected values are all identically 1.

Limit Theorems and Applications

Example

Beverly has a radio that works on a single battery. As soon as the battery fails, Beverly immediately replaces it with a new one. If the lifetime of a battery is uniform between 30 and 60 hours, then, at what rate does Beverly have to change batteries?

Limit Theorems and Applications

Example

Beverly has a radio that works on a single battery. As soon as the battery fails, Beverly immediately replaces it with a new one. If the lifetime of a battery is uniform between 30 and 60 hours, then, at what rate does Beverly have to change batteries?

If we let $N(t)$ be the number of batteries that have failed by time t, we have, by the previous preposition, that the rate at which Beverly replaces batteries is given by

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{1}{\mu} = \frac{1}{45}$$

Example

Customers arrive at a single-server system in accordance with a Poisson process with rate λ, and only enter in the system if the server is idle when she arrives. Assume that the time spent in the system for an entering customer is an RV with distribution G.

(a) What is the rate at which customers enter the system?

(b) What proportion of potential customers actually enter the system?

Example

Assume that the first customer enter the bank at time 0 (the process starts with the arrival of the first customer), and let μ_G be the mean service time. It follows, by the memoryless property of the Poisson distribution, that the mean time between the entering customers is

$$\mu = \mu_G + \frac{1}{\lambda}$$

Hence, the rate at which customers enter is

$$\frac{1}{\mu} = \frac{\lambda}{1 + \lambda\mu_G}$$

Limit Theorems and Applications

Example-continued

The mean time between the entering customers is

$$\mu = \mu_G + \frac{1}{\lambda}$$

Hence, the rate at which customers enter is

$$\frac{1}{\mu} = \frac{\lambda}{1 + \lambda\mu_G}$$

Since the potential customers arrive at a rate λ, the proportion of them entering the system is

$$\frac{\lambda}{\lambda(1 + \lambda\mu_G)} = \frac{1}{1 + \lambda\mu_G}$$

Limit Theorems and Applications

A key element in the proof of elementary renewal theorem is the establishment of the following relationship:

Proposition

$$E[S_{N(t)+1}] = \mu[m(t) + 1]$$

Prove this!

Limit Theorems and Applications

An important theorem is the central limit theorem for renewal processes.

Central Limit Theorem for Renewal Processes

$$\lim_{t\to\infty} P\left\{\frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} < x\right\} = \frac{1}{2\pi}\int_{-\infty}^{x} e^{-x^2/2}dx$$

We can also show that

$$\lim_{t\to\infty} \frac{\text{Var}\,[N(t)]}{t} = \frac{\sigma^2}{\mu^3}$$

Limit Theorems and Applications

Example

An unending number of jobs are processed on 2 machines. The time it takes to process a job on the first machine is a gamma RV with $n = 4$ and $\lambda = 2$ whereas it is uniform between 0 and 4 on the second machine. Approximate the probability that together both machines can process 90 jobs by time $t = 100$.

Limit Theorems and Applications

Example

If we let $N_i(t)$ be the number of jobs that machine i can process by time t, then, $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent renewal processes. We thus have

$$\left.\begin{array}{l} N_1 \sim N(50, 100/8) \\ N_2 \sim N(50, 100/6) \end{array}\right\} \Rightarrow N_1(100) + N_2(100) \sim N\left(100, \frac{175}{6}\right)$$

Limit Theorems and Applications

Example

By letting $N = N_1(100) + N_2(100)$, we can write,

$$\begin{aligned} P\{N > 89.5\} = P\left\{\frac{N-100}{\sqrt{175/6}} > \frac{89.5-100}{\sqrt{175/6}}\right\} &\approx 1 - \Phi\left(\frac{-10.5}{\sqrt{175/6}}\right) \\ &= \Phi\left(\frac{+10.5}{\sqrt{175/6}}\right) \\ &= \Phi(1.944) \\ &= 0.9741 \end{aligned}$$

Renewal Reward Processes

Consider a renewal process $\{N(t), t \geq 0\}$ with time between arrivals $X_n, n \geq 1\}$, and suppose that we receive a reward each time a renewal occurs. Let R_n be the reward earned at the time of the nth renewal. Assume that $R_n, n \geq 1$ are IID, however, R_n may depend on X_n, the length of nth renewal interval. If we let,

$$R(t) = \sum_{n=1}^{N(t)} R(n)$$

then, $R(t)$ represents the total reward earned by time t. Let

$$E(R) = E(R_n) \;\text{ and }\; E(X) = E(X_n)$$

Renewal Reward Processes

Proposition

If $E(R) < \infty$ and $E(X) < \infty$, then,

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{E(R)}{E(X)}$$

and

$$\lim_{t\to\infty} \frac{E[R(t)]}{t} = \frac{E(R)}{E(X)}$$

Regenerative Processes

Consider a stochastic process $\{X_t, t \geq 0\}$ having the property that there exist time points at which the process (probabilistically) restarts itself.

That is, suppose that, with probability 1, there exists a time T_1, such that, the continuation of the process beyond T_1 is a probabilistic replica of the whole process starting at 0. Note that, this property is implies the existence of further times $T_2, T_3, \ldots$ having the same property. Such a stochastic process is known as a *regenerative process*.

Note that, $T_1, T_2, \ldots$ constitute the arrival times of a renewal process, and we say that a cycle is completed every time a renewal occurs.

Regenerative Processes

For example,

(i) A renewal process is regenerative, and T_1 represents the time of the first arrival.

(ii) A recurrent MC is regenerative, and T_1 represents the time of the first transition into the initial state.

Regenerative Process

Proposition

For a regenerative process, the long-run

$$\pi_j = \frac{E(\text{amount of time in } j \text{ during a cycle})}{E(\text{time of a cycle})}$$

Semi-Markov Processes

Consider a process that can be in state i, $i = 1, \ldots, N$ where it remains for a random amount of time with mean μ_i, and then, makes a transition into state j with probability P_{ij}, such a process is called a semi-Markov process. If

$$\mu_i = 1, \quad \forall i$$

then, the semi-Markov process is a Markov chain.

Proportion of Time in Each State

To compute P_i, first consider π_i, which are the unique non-negative solution of

$$\sum_{i=1}^{N} \pi_i = 1$$

and

$$\pi_i = \sum_{i=1}^{N} \pi_j P_{ij}, \quad \forall i$$

We then have

$$P_i = \frac{\pi_i \mu_i}{\sum_{j=1}^{N} \pi_j \mu_j}, \quad \forall i$$

Proportion of Time in Each State

Example

Consider a machine is in one of three states; good, fair and bad. If it is in good condition, it will remain in this condition for a mean time of μ_1 and then go to fair or bad conditions with respective probabilities 3/4 and 1/4. A machine in fair condition will remain in this condition for a mean time of μ_2 and then go to bad condition. A bad machine will be repaired, which takes a mean time of μ_3 and then be in good and fair conditions with respective probabilities 2/3 and 1/3. What proportion of time the machine is the machine in each state?

Proportion of Time in Each State

Example

We have

$$\begin{aligned}
\pi_1 + \pi_2 + \pi_3 &= 1 \\
\pi_1 &= \frac{2}{3}\pi_3 \\
\pi_2 &= \frac{3}{4}\pi_1 + \frac{1}{3}\pi_3 \\
\pi_3 &= \frac{1}{4}\pi_1 + \pi_2
\end{aligned}$$

The solution is then

$$\pi_1 = \frac{4}{15}, \pi_2 = \frac{1}{3}, \pi_3 = \frac{2}{5}$$

Proportion of Time in Each State

Example

We can then compute the proportions as

$$\begin{aligned}
P_1 &= \frac{4\mu_1}{4\mu_1 + 5\mu_2 + 6\mu_3} \\
P_2 &= \frac{5\mu_2}{4\mu_1 + 5\mu_2 + 6\mu_3} \\
P_3 &= \frac{6\mu_3}{4\mu_1 + 5\mu_2 + 6\mu_3}
\end{aligned}$$

The Inspection Paradox

To calculate the distribution of $X_{N(t)+1}$, we condition on the time of the last renewal prior to (or at) time t.

$$P\{X_{N(t)+1} > x\} = E[P\{X_{N(t)+1} > x | S_{N(t)} = t - s\}]$$

It follows that, if $s > x$,

$$P\{X_{N(t)+1} > x | S_{N(t)} = t - s\} = 1$$

and for $s \leq x$,

$$P\{X_{N(t)+1} > x | S_{N(t)} = t - s\} = 1 = \frac{1 - F(X)}{1 - F(s)} \geq 1 - F(x)$$

Taking expectations yields,

$$P\{X_{N(t)+1} > x\} \geq 1 - F(x)$$

Computing the Renewal Function

The determination of

$$F_n(t) = P\{X_1 + X_2 + \ldots + X_n \leq t\}$$

requires the computation of an n-dimensional integral. We can use an efficient algorithm which requires as inputs only one-dimensional integrals.

Let Y be an exponential RV with rate λ, and suppose that Y is independent of the renewal process. We start by determining the $E[N(Y)]$ and start by conditioning on the first renewal.

$$E[N(Y)] = \int_0^\infty E[N(Y) | X_1 = x] f(x) dx$$

where f is interarrival density. We can show that

$$E[N(Y)] = \frac{E[e^{-\lambda X}]}{1 - E[e^{-\lambda X}]}$$

Applications to Patterns

A counting process with independent arrival times $X_1, X_2, \ldots$ is said to be a *delayed* or *general* renewal process if X_1 has a different distribution from the remaining IID RVs.
In other words, a delayed renewal process is a renewal process in which the first inter-arrival time has a different distribution than the others. Note that a delayed renewal process often arise in practice and it is important to note that all of the limiting theorems about $N(t)$ remain valid. For instance,

$$\lim_{t\to\infty} \frac{E[N(t)]}{t} = \frac{1}{\mu}$$

and

$$\lim_{t\to\infty} \frac{\mathrm{Var}\,[N(t)]}{t} = \frac{\sigma^2}{\mu^3}$$

Applications to Patterns

- Patterns of Discrete RVs
- The Expected Time to a Max Run of Distinct Values
- Increasing Run of Continuous RVs

The Insurance Ruin Problem

- Insurance claims according to a Poisson process with rate λ.
- Claim amounts $Y_1, Y_2, \ldots$ are IID RVs with distribution F and density $f(x)$.
- Claim amounts are independent of the arrival times.
- The firm starts with an initial capital of x and receives a income at a constant rate c per unit time.

The probability that the firm's capital ever becomes negative (i.e., the probability of ruin).

$$R(x) = P\left\{\sum_{i=1}^{M(t)} Y_i > x + ct, \exists t \geq 0\right\}$$

The Insurance Ruin Problem

By conditioning on what happens during the first h time units,

$$R(x) = (1 - \lambda h)R(x + ch) + \lambda h E[R(x + ch - Y_1)] + o(h)$$

By organizing, we have

$$\frac{R(x + ch) - R(x)}{ch} = \frac{\lambda}{c}R(x + ch) - \frac{\lambda}{c}E[R(x + ch - Y_1)]\frac{1}{c} + \frac{o(h)}{h}$$

By letting $h \to 0$, we have

$$\frac{dR(x)}{dx} = \frac{\lambda}{c}R(x) - \frac{\lambda}{c}E[R(x - Y_1)]$$

The Insurance Ruin Problem

Since $R(u) = 1$ when $u < 0$, we can write the preceding as

$$\begin{aligned}\frac{dR(x)}{dx} &= \frac{\lambda}{c}R(x) - \frac{\lambda}{c}\int_0^x R(x-y)f(y)dy - \frac{\lambda}{c}\int_x^\infty f(y)dy \\ &= \frac{\lambda}{c}R(x) - \frac{\lambda}{c}\int_0^x R(x-y)f(y)dy - \frac{\lambda}{c}[1-F(x)]\end{aligned}$$

We can show that the preceding satisfies the following equation.

$$R(x) = R(0) + \frac{\lambda}{c}\int_0^x R(x-y)f(y)[1-F(y)]dy - \frac{\lambda}{c}\int_0^x [1-F(y)]dy$$

The End

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