# Continuous Time Markov Chains <br> Stochastic Processes - Lecture Notes 

## Fatih Cavdur

to accompany
Introduction to Probability Models by Sheldon M. Ross

Fall 2015

## Outline

Introduction

Continuous-Time Markov Chains

Birth and Death Processes

The Transition Probability Function

Limiting Probabilities

## Introduction

In this chapter we consider a class of probability models that has a wide variety of applications in the real world. The members of this class are the continuous-time analogs of the Markov chains of Chapter 4 and as such are characterized by the Markovian property that, given the present state, the future is independent of the past. One example of a continuous-time Markov chain is the Poisson process of Chapter 5.

## Continuous-Time Markov Chains

Suppose that we have a continuous-time stochastic process
$\{X(t), t \geq 0\}$ taking on values in the set of non-negative integers.
We can say that $\{X(t), t \geq 0\}$ is a continuous-time Markov chain
(CT-MC), if, for all $s, t \geq 0$ and non-negative integers
$i, j, x(u), 0 \leq u<s$
$P\{X(t+s)=j \mid X(s)=i, X(u)=x(u)\}=P\{X(t+s)=j \mid X(s)=i\}$

## Continuous-Time Markov Chains

In addition, if $P\{X(t+s)=j \mid X(s)=i\}$ is independent of $s$, then, the CT-MC is said to have stationary or homogenous transition probabilities.
We assume that all MCs in this section have stationary transition probabilities.

## Continuous-Time Markov Chains

Another definition of a CT-MC is a stochastic process having the properties that each time it enters state $i$

- the amount of time it spends in that state before making a transition into a different state is exponentially distributed with mean $1 / v_{i}$, and
- when the process leaves state $i$, it next enters state $j$ with some probability, $P_{i j}$ where

$$
\begin{array}{rr}
P_{i i}=0, & \forall i \\
\sum_{j} P_{i j}=1, & \forall i
\end{array}
$$

## Continuous-Time Markov Chains

In other words, a CT-MC is a stochastic process that moves from state to state in accordance with a (discrete-time) Markov chain, but is such that the amount of time it spends in each state, before proceeding to the next state, is exponentially distributed. In addition, the amount of time the process spends in state $i$ and the next state it enters must be independent RVs

## Example

(A Shoeshine Shop) Consider a shoeshine establishment consisting of two chairs-chair 1 and chair 2. A customer upon arrival goes initially to chair 1 where his shoes are cleaned and polish is applied. After this is done the customer moves on to chair 2 where the polish is buffed. The service times at the two chairs are assumed to be independent random variables that are exponentially distributed with respective rates $\mu_{1}$ and $\mu_{2}$, and the customers arrive in accordance with a Poisson process with rate $\lambda$. We also assume that a potential customer will enter the system only if both chairs are empty.

## Example

We can analyze this system as a CT-MC with a state space

| State | Description |
| :---: | :--- |
| 0 | system is empty |
| 1 | a customer is in chair 1 |
| 2 | a customer is in chair 2 |

Example
We then have

$$
v_{0}=\lambda, \quad v_{1}=\mu_{1}, \quad v_{2}=\mu_{2}
$$

and

$$
P_{01}=P_{12}=P_{20}=1
$$

## Birth and Death Processes

Consider a system whose state at any time is represented by the number of people in the system at that time. Suppose that whenever there are $n$ people in the system

- new arrivals enter the system at an exponential rate $\lambda_{n}$, and
- people leave the system at an exponential rate $\mu_{n}$.

Such a system is called a birth and death (arrival and departure) process (BDP or ADP), and the parameters $\left\{\lambda_{n}\right\}_{n=0}^{\infty}$ and $\left\{\mu_{n}\right\}_{n=1}^{\infty}$ are called birth (arrival) and death (departure) rates, respectively.

## Birth and Death Processes

A BDP is thus a CT-MC with states $\{0,1, \ldots\}$ for which transitions from state $n$ may go only to state $n-1$ or state $n+1$, and

$$
v_{0}=\lambda_{0} ; \quad v_{i}=\lambda_{i}+\mu_{i}, \quad i>0
$$

and

$$
P_{01}=1 ; \quad P_{i, i+1}=\frac{\lambda_{i}}{\lambda_{i}+\mu_{i}}, i>0 ; \quad P_{i, i-1}=\frac{\mu_{i}}{\lambda_{i}+\mu_{i}}, i>0
$$

## Example (Poisson Process)

Consider a BDP for which

$$
\begin{aligned}
& \mu_{n}=0, \quad \forall n \geq 0 \\
& \lambda_{n}=\lambda, \quad \forall n \geq 0
\end{aligned}
$$

This is process in which no departures occur, and the time between arrivals is exponentially distributed with mean $1 / \lambda$. Hence, this is the Poisson process.

## Birth and Death Process

A BDP for which $\mu_{n}=0, \forall n$ is is said to be a pure birth process, and a BDP for which $\lambda_{n}=0, \forall n$ is said to be pure death process.

## Example

A model in which

$$
\begin{aligned}
& \mu_{n}=n \mu, \quad n \geq 1 \\
& \lambda_{n}=n \lambda+\theta, \quad n \geq 0
\end{aligned}
$$

is called a linear growth process with immigration and used to study biological systems. If $X(t)$ is the population size at time $t$, and if we assume that $X(0)=i$ and let

$$
M(t)=E[X(t)]
$$

## Example

Derive and solve a differential equation to determine $M(t)$ :

$$
\begin{aligned}
M(t+h) & =E[X(t+h)] \\
& =E\{E[X(t+h) \mid X(t)]\}
\end{aligned}
$$

We can write
$X(t+h)=\left\{\begin{array}{cl}X(t+1), & \text { w.p. }[\theta+X(t) \lambda] h+o(h) \\ X(t-1), & \text { w.p. } X(t) \mu h+o(h) \\ X(t), & \text { w.p. } 1-[\theta+X(t) \lambda+X(t) \mu] h+o(h)\end{array}\right.$

## Example

We then have

$$
\begin{aligned}
E[X(t+h) \mid X(t)] & =X(t)+[\theta+X(t) \lambda-X(t) \mu] h+o(h) \\
E\{E[X(t+h) \mid X(t)]\} & =E\{X(t)+[\theta+X(t) \lambda-X(t) \mu] h+o(h)\} \\
M(t+h) & =M(t)+(\lambda-\mu) M(t) h+\theta h+o(h) \\
\frac{M(t+h)-M(t)}{h} & =(\lambda-\mu) M(t)+\theta+\frac{o(h)}{h} \\
\lim _{h \rightarrow 0}\left[\frac{M(t+h)-M(t)}{h}\right] & =\lim _{h \rightarrow 0}\left[(\lambda-\mu) M(t)+\theta+\frac{o(h)}{h}\right] \\
\frac{d M(t)}{d t} & =(\lambda-\mu) M(t)+\theta
\end{aligned}
$$

## Example

If we define

$$
h(t)=(\lambda-\mu) M(t)+\theta \Rightarrow \frac{d h(t)}{d t}=(\lambda-\mu) \frac{d M(t)}{d t}
$$

We can hence write

$$
\begin{aligned}
\frac{h^{\prime}(t)}{\lambda-\mu}=h(t) \Rightarrow \frac{h^{\prime}(t)}{h(t)} & =\lambda-\mu \\
\log [h(t)] & =(\lambda-\mu) t+c \\
h(t) & =K e^{(\lambda-\mu) t}
\end{aligned}
$$

## Example

In terms of $M(t)$,

$$
\theta+(\lambda-\mu) M(t)=K e^{(\lambda-\mu) t}
$$

To find $K$, we use that $M(0)=i$ and for $t=0$,

$$
\theta+(\lambda-\mu) i=K \Rightarrow M(t)=\frac{\theta}{\lambda-\mu}\left[e^{(\lambda-\mu) t}-1\right]+i e^{(\lambda-\mu) t}
$$

Note that, we have assumed that $\lambda \neq \mu$. If $\lambda=\mu$,

$$
\frac{d M(t)}{d t}=\theta \Rightarrow M(t)=\theta t+i
$$

## Example

Suppose that customers arrive at a single-server service station in accordance with a Poisson process having rate $\lambda$, and the successive service times are assumed to be independent exponential RVs with mean $1 / \mu$. This is known as the $M / M / 1$ queuing system where the first and second $M$ refer to the Poisson arrivals and the exponential service times, both are Markovian, and 1 refers to the number of servers.

## Example

If we let $X(t)$ be the number of customers in the system at time $t$, then, $\{X(t), t \geq 0\}$ is a BDP process with with

$$
\begin{array}{ll}
\mu_{n}=\mu, & n \geq 1 \\
\lambda_{n}=\lambda, & n \geq 0
\end{array}
$$

## Example

For a multi-server exponential queuing system with $s$ servers, if we let $X(t)$ be the number of customers in the system at time $t$, then, $\{X(t), t \geq 0\}$ is a BDP process with with

$$
\mu_{n}=\left\{\begin{array}{cc}
n \mu, & 1 \leq n \leq s \\
s \mu, & n>s
\end{array}\right.
$$

and

$$
\lambda_{n}=\lambda, \quad n \geq 0
$$

## The Transition Probability Function

We let

$$
P_{i j}(t)=P\{X(t+s)=j \mid X(s)=i\}
$$

be the probability that a process presently in state $i$ will be in state $j$ after time $t . P_{i j}(t)$ are often called the transition probabilities of the CT-MC.

## The Transition Probability Function

## Proposition

For a pure birth process with $\lambda_{i} \neq \lambda_{j}$ when $i \neq j$, we have

$$
P_{i j}(t)=\sum_{k=i}^{j} e^{-\lambda_{k} t} \prod_{\substack{r=i \\ r \neq k}}^{j} \frac{\lambda_{r}}{\lambda_{r}-\lambda_{k}}-\sum_{k=i}^{j-1} e^{-\lambda_{k} t} \prod_{\substack{r=i \\ r \neq k}}^{j-1} \frac{\lambda_{r}}{\lambda_{r}-\lambda_{k}}, \quad i<j
$$

and

$$
P_{i j}(t)=e^{-\lambda_{i} t}
$$

## Kolmogorov's Backward Equations

Theorem: Kolmogorov's Backward Equations
For all states $i, j$ and times $t \geq 0$,

$$
P_{i j}^{\prime}(t)=\sum_{k \neq i} q_{i k} P_{k j}(t)-v_{i} P_{i j}(t)
$$

## Example

The backward equations for the pure birth process are

$$
P_{i j}^{\prime}(t)=\lambda_{i} P_{i+1, j}(t)-\lambda_{i} P_{i j}(t)
$$

Example: Backward Equations for the BDP

$$
\begin{aligned}
& P_{0 j}^{\prime}(t)=\lambda_{0} P_{1 j}(t)-\lambda_{0} P_{0 j}(t) \\
& P_{i j}^{\prime}(t)=\left(\lambda_{i}+\mu_{i}\right)\left[\frac{\lambda_{i}}{\lambda_{i}+\mu_{i}} P_{i+1, j}(t)+\frac{\mu_{i}}{\lambda_{i}+\mu_{i}} P_{i-1, j}(t)\right]-\left(\lambda_{i}+\mu_{i}\right) P_{i j}(t) \\
& \text { or }
\end{aligned}
$$

$$
\begin{gathered}
P_{0 j}^{\prime}(t)=\lambda_{0}\left[P_{1 j}(t)-P_{0 j}(t)\right] \\
P_{i j}^{\prime}(t)=\lambda_{i} P_{i+1, j}(t)+\mu_{i} P_{i-1, j}(t)-\left(\lambda_{i}+\mu_{i}\right) P_{i j}(t)
\end{gathered}
$$

## Kolmogorov's Forward Equations

Theorem: Kolmogorov's Forward Equations
Under suitable regularity conditions,

$$
P_{i j}^{\prime}(t)=\sum_{k \neq j} q_{k j} P_{i k}(t)-v_{j} P_{i j}(t)
$$

## Kolmogorov's Backward Equations

For the pure birth process, we have

$$
P_{i j}^{\prime}(t)=\lambda_{j-1} P_{i, j-1}(t)-\lambda_{j} P_{i j}(t)
$$

Noting that $P_{i j}(t)=0$ whenever $j<i$, we can write

$$
P_{i i}^{\prime}(t)=\lambda_{i} P_{i, i}(t)
$$

and

$$
P_{i j}^{\prime}(t)=\lambda_{j-1} P_{i, j-1}(t)-\lambda_{j} P_{i j}(t), \quad j \geq i+1
$$

## Kolmogorov's Backward Equations

For the BDP, we have

$$
\begin{gathered}
P_{i 0}^{\prime}(t)=\sum_{k \neq 0} q_{k 0} P_{i k}(t)-\lambda_{0} P_{i 0}(t) \\
=\mu_{1} P_{i, 1}(t)-\lambda_{0} P_{i 0}(t) \\
P_{i j}^{\prime}(t)=\sum_{k \neq 0} q_{k j} P_{i k}(t)-\left(\lambda_{j}+\mu_{j}\right) P_{i j}(t) \\
=\lambda_{j-1} P_{i, j-1}(t)-\left(\lambda_{j}+\mu_{j}\right) P_{i j}(t)
\end{gathered}
$$

## Limiting Probabilities

In analogy with a basic result in discrete-time Markov chains, the probability that a continuous-time Markov chain will be in state $j$ at time $t$ often converges to a limiting value that is independent of the initial state. If we call this value $P_{j}$, then,

$$
P_{j} \equiv \lim _{t \rightarrow \infty} P_{i j}(t)
$$

where we assume that the limit exists and is independent of the initial state $i$.

## Limiting Probabilities

To derive $P_{j}$ using the forward equations, we can write

$$
P_{i j}^{\prime}(t)=\sum_{k \neq j} q_{k j} P_{i k}(t)-v_{j} P_{i j}(t)
$$

and

$$
\begin{aligned}
\lim _{t \rightarrow \infty} P_{i j}^{\prime}(t) & =\lim _{t \rightarrow \infty}\left[\sum_{k \neq j} q_{k j} P_{i k}(t)-v_{j} P_{i j}(t)\right] \\
& =\sum_{k \neq j} q_{k j} P_{k}-v_{j} P_{j}
\end{aligned}
$$

## Limiting Probabilities

We note that, since $P_{i j}^{\prime}(t)$ converges to 0 , we can write

$$
0=\sum_{k \neq j} q_{k j} P_{k}-v_{j} P_{j} \Rightarrow v_{j} P_{j}=\sum_{k \neq j} q_{k j} P_{k}
$$

We can then use the following to find the limiting probabilities:

$$
\begin{aligned}
v_{j} P_{j} & =\sum_{k \neq j} q_{k j} P_{k}, \forall j \\
1 & =\sum_{j} P_{j}
\end{aligned}
$$

## Limiting Probabilities for the BDP

We can write,
State 0: $\quad \lambda_{0} P_{0}=\mu_{1} P_{1}$
State 1: $\left(\lambda_{1}+\mu_{1}\right) P_{1}=\mu_{2} P_{2}+\lambda_{0} P_{0}$
State 2: $\left(\lambda_{2}+\mu_{2}\right) P_{2}=\mu_{3} P_{3}+\lambda_{1} P_{1}$
State $n:\left(\lambda_{n}+\mu_{n}\right) P_{n}=\mu_{n+1} P_{n+1}+\lambda_{n-1} P_{n-1}$

Limiting Probabilities for the BDP
By organizing, we obtain

$$
\begin{aligned}
\lambda_{0} P_{0} & =\mu_{1} P_{1} \\
\lambda_{1} P_{1} & =\mu_{2} P_{2} \\
\lambda_{2} P_{2} & =\mu_{3} P_{3} \\
\ldots & =\ldots \\
\lambda_{n} P_{n} & =\mu_{n+1} P_{n+1}
\end{aligned}
$$

Limiting Probabilities for the BDP
Solving in terms of $P_{0}$,

$$
\begin{aligned}
P_{1} & =\frac{\lambda_{0}}{\mu_{1}} P_{0} \\
P_{2} & =\frac{\lambda_{1} \lambda_{0}}{\mu_{2} \mu_{1}} P_{0} \\
P_{3} & =\frac{\lambda_{2} \lambda_{1} \lambda_{0}}{\mu_{3} \mu_{2} \mu_{1}} P_{0} \\
\ldots & =\ldots \\
P_{n} & =\frac{\lambda_{n-1} \ldots \lambda_{1} \lambda_{0}}{\mu_{n} \ldots \mu_{2} \mu_{1}} P_{0}
\end{aligned}
$$

## Limiting Probabilities for the BDP

Using that $\sum_{n=0}^{\infty} P_{n}=1$, we obtain

$$
1=P_{0}+\sum_{n=1}^{\infty} \frac{\lambda_{n-1} \ldots \lambda_{1} \lambda_{0}}{\mu_{n} \ldots \mu_{2} \mu_{1}} P_{0} \Rightarrow P_{0}=\frac{1}{1+\sum_{n=1}^{\infty} \frac{\lambda_{n-1} \ldots \lambda_{1} \lambda_{0}}{\mu_{n} \ldots \mu_{2} \mu_{1}}}
$$

and so

$$
P_{n}=\frac{\lambda_{0} \lambda_{1} \ldots \lambda_{n-1}}{\mu_{1} \mu_{2} \ldots \mu_{n} \frac{1}{1+\sum_{n=1}^{\infty} \frac{\lambda_{n-1} \ldots \lambda_{1} \lambda_{0}}{\mu_{n} \ldots \mu_{2} \mu_{1}}}}, \quad n \geq 1
$$

## Limiting Probabilities for the BDP

The foregoing equations also show us the necessary conditions for the limiting probabilities to exist:

$$
\sum_{n=1}^{\infty} \frac{\lambda_{n-1} \ldots \lambda_{1} \lambda_{0}}{\mu_{n} \ldots \mu_{2} \mu_{1}}<\infty
$$

This condition also might be shown to be sufficient.

## Limiting Probabilities

Let us reconsider the shoeshine shop of Example 6.1, and determine the proportion of time the process is in each of the states $0,1,2$. Because this is not a birth and death process (since the process can go directly from state 2 to state 0 ), we start with the balance equations for the limiting probabilities.

## Limiting Probabilities for the BDP

We can write,

$$
\begin{array}{ll}
\text { State 0: } & \lambda_{0} P_{0}=\mu_{2} P_{2} \\
\text { State 1: } & \mu_{1} P_{1}=\lambda P_{0} \\
\text { State 2: } & \mu_{2} P_{2}=\mu_{1} P_{1}
\end{array}
$$

Limiting Probabilities for the BDP
We then write,

$$
P_{1}=\frac{\lambda}{\mu_{1}} P_{0}, \quad P_{2}=\frac{\lambda}{\mu_{2}} P_{0}
$$

and

$$
\sum_{i=0}^{2} P_{i}=1 \Rightarrow P_{0}=\frac{\mu_{1} \mu_{2}}{\mu_{1} \mu_{2}+\lambda\left(\mu_{1}+\mu_{2}\right)}
$$

and so

$$
P_{1}=\frac{\lambda \mu_{2}}{\mu_{1} \mu_{2}+\lambda\left(\mu_{1}+\mu_{2}\right)}, \quad P_{1}=\frac{\lambda \mu_{1}}{\mu_{1} \mu_{2}+\lambda\left(\mu_{1}+\mu_{2}\right)}
$$

Thanks! Questions?

