Exponential Distribution and Poisson Process

Stochastic Processes - Lecture Notes

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to accompany
Introduction to Probability Models
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Outline

Introduction

In a mathematical model for a real-world phenomenon, we must make enough simplifying assumptions to enable us to handle the mathematics, but not so-many that the mathematical model no longer resembles the real-world phenomenon. One of these assumptions is to assume that certain RVs are exponentially distributed. A counting process that is related to the exponential distribution, namely, the Poisson process is also considered in this chapter.

Exponential Distribution

An RV X is said to have an exponential distribution with parameter λ, $\lambda > 0$, if its PDF is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

or, if its CDF is given by

$$F(X) = \int_{-\infty}^{x} f(y)dy = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & 0 \end{cases}$$

Exponential Distribution

The mean of the exponential distribution is given by

$$\begin{aligned} E(X) &= \int_{-\infty}^{+\infty} x f(x) dx \\ &= \int_{0}^{\infty} \lambda x e^{-\lambda x} dx \\ &= -x e^{-\lambda x}\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx \\ &= \frac{1}{\lambda} \end{aligned}$$

Exponential Distribution

Second moment of the RV X is

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{+\infty} x^2 f(x) dx \\ &= \int_{0}^{\infty} \lambda x^2 e^{-\lambda x} dx \\ &= -x^2 e^{-\lambda x}\Big|_0^{\infty} + \int_0^{\infty} 2x e^{-\lambda x} dx \\ &= \frac{2}{\lambda^2} \end{aligned}$$

Exponential Distribution

Variance of the RV X is

$$\begin{aligned}
\text{Var }(X) &= E(X^2) - [E(X)]^2 \\
&= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \\
&= \frac{1}{\lambda^2}
\end{aligned}$$

Exponential Distribution

The MGF of X is

$$\begin{aligned}
\phi(t) &= E(e^{tX}) \\
&= \int_0^\infty \lambda e^{tx} e^{-\lambda x} \\
&= \frac{\lambda}{\lambda - t}, \quad t < \lambda
\end{aligned}$$

Exponential Distribution

The moments of X are obtained using the MGF. For example,

$$\begin{aligned}
E(X) &= \frac{d\phi(t)}{dt}\Big|_{t=0} \\
&= \frac{d}{dt}\left(\frac{\lambda}{\lambda - t}\right)\Big|_{t=0} \\
&= \frac{\lambda}{(\lambda - t)^2}\Big|_{t=0} \\
&= \frac{1}{\lambda}
\end{aligned}$$

Exponential Distribution

The moments of X are obtained using the MGF. For example,

$$\begin{aligned}
E(X^2) &= \frac{d^2\phi(t)}{dt}\Big|_{t=0} \\
&= \frac{d^2}{dt}\left(\frac{\lambda}{\lambda - t}\right)\Big|_{t=0} \\
&= \frac{2\lambda}{(\lambda - t)^3}\Big|_{t=0} \\
&= \frac{2}{\lambda}
\end{aligned}$$

Exponential Distribution

The variance can then also be computed as

$$\begin{aligned}\text{Var }(X) &= E(X^2) - [E(X)]^2\\ &= \frac{2}{\lambda} - \frac{1}{\lambda}\\ &= \frac{1}{\lambda}\end{aligned}$$

Properties of the Exponential Distribution

An RV X is said to be without memory or memoryless if

$$P\{X > s + t | X > t\} = P\{X > s\}, \quad \forall s, t \geq 0$$

or equivalently,

$$\frac{P\{X > s + t, X > t\}}{P\{X > t\}} = P\{X > s\}$$

and, hence,

$$P\{X > s + t\} = P\{X > s\}P\{X > t\}$$

Properties of the Exponential Distribution

Hence, X is said to be without memory or memoryless if

$$P\{X > s + t\} = P\{X > s\}P\{X > t\}$$

and when X is exponentially distributed,

$$P\{X > s + t\} = e^{-\lambda(s+t)} = e^{-\lambda s}e^{-\lambda t} = P\{X > s\}P\{X > t\}$$

Example

Suppose that the amount of time one spends in a bank is exponentially distributed with with mean 10 minutes. What is the probability that a customer will spend more than 15 minutes in the bank? What is the probability that a customer will spend more than 15 minutes in the bank given that she is still in the bank after 10 minutes?

Example

The probability that a customer will spend more than 15 minutes in the bank is

$$P\{X > 15\} = 1 - F(15) = e^{-15\lambda} = e^{-3/2}$$

which is approximately equal to 0.220. The probability that a customer will spend more than 15 minutes in the bank given that she is still in the bank after 10 minutes is

$$P\{X > 15|X > 10\} = P\{X > 5\} = 1 - F(5) = e^{-5\lambda} = e^{-1/2}$$

which is approximately 0.604.

Properties of Exponential Distribution

Let $X_1, \ldots, X_n$ be IID exponential RVs with rate λ. It follows that the distribution of $X_1, \ldots, X_n$ is gamma with parameters n and λ. By using induction, we can prove that. Since nothing to do for $n = 1$, we assume that $X_1 + \ldots + X_{n-1}$ has a gamma distribution with $n - 1$ and λ.

Properties of Exponential Distribution

Assume that $X_1 + \ldots + X_{n-1} \sim$ gamma $(n-1, \lambda)$:

$$f_{X_1+\ldots+X_{n-1}}(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-2}}{(n-2)!}$$

and hence,

$$\begin{aligned} f_{X_1+\ldots+X_n}(t) &= \int_0^\infty f_{x_n}(t-s) f_{X_1+\ldots+X_{n-1}}(s) ds \\ &= \int_0^t \lambda e^{-\lambda(t-s)} \frac{\lambda e^{-\lambda s}(\lambda s)^{n-2}}{(n-2)!} ds \\ &= \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} \end{aligned}$$

Properties of Exponential Distribution

Assume that X_1 and X_2 are exponential with λ_1 and λ_2. The probability that $X_1 < X_2$ can be computed by conditioning on X_1 as

$$\begin{aligned} P\{X_1 < X_2\} &= \int_0^\infty P\{X_1 < X_2 | X_1 = x\} \lambda_1 e^{-\lambda_1 x} dx \\ &= \int_0^\infty P\{x < X_2\} \lambda_1 e^{-\lambda_1 x} dx \\ &= \int_0^\infty e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x} dx \\ &= \int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2)x} dx \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2} \end{aligned}$$

Counting Process

A stochastic process $\{N(t), t \geq 0\}$ is said to be a counting process if $N(t)$ represents the the total number of "events" that occur by time t. A counting process $N(t)$ must satisfy

- $N(t) \geq 0$.
- $N(t)$ is integer valued.
- If $s < t$, then, $N(s) \leq N(t)$.
- For $s < t$, $N(t) - N(s)$ equals the number of events occur in the interval $(s, t]$.

Counting Process

A stochastic process $\{N(t), t \geq 0\}$ is a counting process if $N(t)$ represents

- the number of persons who enter a store at or prior to time t (event: a person entering the store)
- the number of people who were born by time t (event: born of a child)

Counting Process

- A counting process is said to posses independent increments if the number of events that occur in disjoint time intervals are independent.
- A counting process is said to posses stationary increments if the distribution of the number of events that occur in any interval of time depends only on the length of the interval.

Poisson Process

Definition

The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate λ, $\lambda > 0$, if

(i) $N(0) = 0$.

(ii) The process has independent increments.

(iii) The number of events in any interval of length t is Poisson distributed with mean λt. That is, for all $s, t \geq 0$,

$$P\{N(t+s) - N(s) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0, 1, \ldots$$

Poisson Process

Definition

The function $f(\bullet)$ is said to be $o(h)$ if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0$$

Poisson Process

The function $f(x) = x^2$ is $o(h)$ since

$$\lim_{h \to 0} \frac{f(h)}{h} = \lim_{h \to 0} \frac{h^2}{h} = 0$$

The function $f(x) = x$ is not $o(h)$ since

$$\lim_{h \to 0} \frac{f(h)}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

Poisson Process

Alternative Definition

The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate λ, $\lambda > 0$, if

(i) $N(0) = 0$.

(ii) The process has independent increments.

(iii) $P\{N(h) = 1\} = \lambda h + o(h)$

(iv) $P\{N(h) \geq 2\} = o(h)$

Inter-Arrival and Waiting Time Distributions

Let T_n be the time between the $n-1$st and and the nth event. To define the distribution of $\{T_n, n = 1, 2, \ldots\}$, note that

$$\begin{aligned} T_1 > t \Leftrightarrow N(t) = 0 \Rightarrow P\{T_1 > t\} &= P\{N(t) = 0\} \\ &= e^{-\lambda t} \Rightarrow T_1 \sim \text{ exponential } (\lambda) \end{aligned}$$

Inter-Arrival and Waiting Time Distributions

Also note that

$$P\{T_2 > t\} = E(P\{T_2 > t | T_1\})$$

and

$$\begin{aligned} P\{T_2 > t | T_1 = s\} &= P\{0 \text{ events in } (s, s+t] | T_1 = s\} \\ &= P\{0 \text{ events in } (s, s+t]\} \\ &= e^{-\lambda t} \Rightarrow T_2 \sim \text{ exponential } (\lambda) \end{aligned}$$

by independence and stationary increments.

Inter-Arrival and Waiting Time Distributions

Hence, we conclude that T_2 is also an exponential RV with rate λ and that T_2 is independent of T_1. Repeating the same argument yields the following:

Proposition

$T_n, n = 1, 2, \ldots$ are IID exponential RVs with rate λ.

Inter-Arrival and Waiting Time Distributions

S_n, the arrival time of the nth event, defined as

$$S_n = \sum_{i=1}^{n} T_i, \quad n \geq 1$$

We can show that $S_n \sim$ gamma (n, λ).

Inter-Arrival and Waiting Time Distributions

The counting process $\{N(t), t \geq 0\}$ is said to be non-homogenous Poisson process with intensity function $\lambda(t)$, $t \geq 0$, if

- $N(t) = 0$
- $\{N(t), t \geq 0\}$ has independent increments
- $P\{N(t+h) - N(t) \geq 2\} = o(h)$
- $P\{N(t+h) - N(t) = 1\} = \lambda(t)h + o(h)$

Inter-Arrival and Waiting Time Distributions

Consider a hot dog stand that opens at 8 A.M. From 8 until 11 A.M. customers arrive, on the average, at a steadily increasing rate that starts with an initial rate of 5 customers per hour at 8 A.M. and reaches a maximum of 20 customers per hour at 11 A.M. From 11 A.M. until 1 P.M. the (average) rate remains constant at 20 customers per hour, and then, drops steadily from 1 P.M. until closing time at 5 P.M. at which time it has the value of 12 customers per hour. If we assume that the numbers of customers arriving during disjoint time periods are independent, then, what is the probability that no customers arrive between 8:30 A.M. and 9:30 A.M. on Monday morning? What is the expected number of arrivals in this period?

Inter-Arrival and Waiting Time Distributions

By assuming a non-homogenous Poisson process with intensity function $\lambda(t)$ defined as

$$\lambda(t) = \begin{cases} 5 + 5t, & 0 \leq t \leq 3 \\ 20, & 3 \leq t \leq 5 \\ 20 - 2(t-5), & 5 \leq t \leq 9 \end{cases}$$

and

$$\lambda(t) = \lambda(t-9), \quad t > 9$$

Inter-Arrival and Waiting Time Distributions

We can also write it as

$$\lambda(t) = \begin{cases} 0, & 0 \le t \le 8 \\ 5 + 5(t-8), & 8 \le t \le 11 \\ 20, & 11 \le t \le 13 \\ 20 - 2(t-13), & 13 \le t \le 17 \\ 0, & 17 \le t \le 24 \end{cases}$$

and

$$\lambda(t) = \lambda(t-24), \quad t > 24$$

Inter-Arrival and Waiting Time Distributions

As the number of arrivals between 8.30 A.M. and 9.30 A.M. will be Poisson with mean $m(3/2) - m(1/2)$ in the first representation. We then have the probability that this number is zero is

$$\exp\left\{-\int_{1/2}^{3/2}(5+5t)dt\right\} = e^{-10}$$

and the mean number of arrivals is

$$\int_{1/2}^{3/2}(5+5t)dt = 10$$

Thanks! Questions?