Introduction to Probability Theory

Stochastic Processes - Lecture Notes

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to accompany
Introduction to Probability Models
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Outline

Probability Models

- Any realistic model of a real-world phenomenon must take into account the possibility of randomness.
- In general, the quantities we are interested in will not be predictable in advance but, rather, will exhibit an inherent variation that should be taken into account by the model.
- This is usually accomplished by allowing the model to be probabilistic in nature.
- Such a model is, naturally enough, referred to as a probability model.

Sample Space

The set of all outcomes of an experiment is known as the sample space, and is denoted by S. Some examples are

1. Flipping a coin: $S = \{h, t\}$.
2. Rolling a die: $S = \{1, 2, \ldots, 6\}$
3. Flipping two coins: $S = \{(h, h), (h, t), (t, h), (t, t)\}$
4. Rolling two dice: $S = \{(i, j) | 1 \leq i, j \leq 6\}$
5. Measuring the life of a car: $S = [0, \infty)$

Events

Any subset E of the sample space S is known as an event. Some examples are

1. $E = \{h\}$.
2. $E = \{1, 2, 3\}$, $F = \{3, 4, 5\}$
3. $E = \{(h, h), (h, t)\}$
4. $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$
5. $S = (1, 10)$

Union, Intersection, Compliment

- For any two events E and F, the *union* of the events consist of all outcomes that are either in E or in F or in both, and is denoted by $E \cup F$.
- For any two events E and F, the *intersection* of the events consist of all outcomes that are both in E and in F, and is denoted by $E \cap F$ or EF.
- If $EF = \Phi$, then, E and F are said to be *mutually exclusive*.
- For an event E, the compliment of the event consist of all outcomes in sample space S that are not in E, and denoted by E^c.

Probability of an Event

We refer to $P(E)$ as the probability of event E if it satisfies the following conditions:

- $0 \leq P(E) \leq 1$
- $P(S) = 1$
- For any sequence of events $E_1, E_2, \ldots$ that are mutually exclusive

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Example

- In the die tossing example, if all outcomes are equally likely, we have

$$P(\{1\}) = P(\{2\}) = \ldots = P(\{6\}) = \frac{1}{6}$$

- The probability of getting an even number is given by

$$P(\{2,4,6\}) = P(\{2\}) = P(\{4\}) = P(\{6\}) = \frac{1}{2}$$

Example

If we toss two coins and suppose that each outcome is equally likely, and let the events E and F defined as the first coin falls heads and the second coin falls heads, respectively. That is, $E = \{(h,h),(h,t)\}$ and $F = \{(h,h),(t,h)\}$ The probability that either the first or the second coin falls heads is then given by

$$
\begin{aligned}
P(E \cup F) &= P(E) + P(F) - P(EF) \\
&= \frac{1}{2} + \frac{1}{2} - P\{(h,h)\} \\
&= 1 - \frac{1}{4} \\
&= \frac{3}{4} = P\{(h,h),(h,t),(t,h)\}
\end{aligned}
$$

Union of More than 2 Events

We can compute the probability of the union of 3 events as

$$
\begin{aligned}
P(E \cup F \cup G) &= P[(E \cup F) \cup G] \\
&= P(E \cup F) + P(G) - P[(E \cup F)G] \\
&= P(E) + P(F) - P(EF) + P(G) - P(EG \cup FG) \\
&= P(E) + P(F) - P(EF) + P(G) \\
&- P(EG) - P(FG) + P(EGFG) \\
&= P(E) + P(F) + P(G) \\
&- P(EF) - P(EG) - P(FG) + P(EFG)
\end{aligned}
$$

Union of More than 2 Events

In general, we can show that

$$\begin{aligned} P(E_1 \cup E_2 \ldots \cup E_n) &= \sum_i P(E_i) \\ &- \sum_{i<j} P(E_i E_j) + \sum_{i<j<k} P(E_i E_j E_k) - \ldots \\ &+ (-1)^{n+1} P(E_1 E_2 \ldots E_n) \end{aligned}$$

Conditional Probabilities

If the event F occurs, then, in order for E to occur it is necessary for the actual occurrence to be a point in both E and F, that is, it must be in EF. Now, since we know that F has occurred, it follows that F becomes our new sample space and hence, the probability that the event EF occurs will equal the probability of EF relative to the probability of F. That is,

$$P(E|F) = \frac{P(EF)}{P(F)}$$

We note that $P(F) > 0$ is required for the above equation to be well-defined.

Example 1.4

10 cards are numbered from 1 to 10 and placed in a hat. Someone draw a card from the hat. If she says the number is at least 5, then, what is the conditional probability that it is 10?

We let E be the event that the number is 10 and F be the event that the number is at least 5. We then need to compute the probability $P(E|F)$ as

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\frac{1}{10}}{\frac{6}{10}} = \frac{1}{6}$$

Example 1.5

A family has 2 children. What is the conditional probability that both are boys given that at least one of them is a boy?

We let E be the event that the both children are boys and F be the event that the number is at least one of them is a boy. We then need to compute the probability $P(E|F)$ as

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{P(\{(b,b)\})}{P\{(b,b),(b,g),(g,b)\}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

Example 1.7

An urn contains 7 black and 5 white balls. We draw 2 balls without replacement. What is the probability that both are black?

Let F and E be the events that the first and second balls are black, respectively. The probability that the second ball is black *given that the first ball is black* is $P(E|F) = 6/11$. Since $P(F) = 7/12$, we have

$$P(EF) = P(F)P(E|F) = \frac{7}{12}\frac{6}{11} = \frac{7}{22}$$

Example 1.8

Suppose that 3 men at a party throw their hats into the center of the room. The hats are somehow mixed up, and then, each man randomly selects a hat. What is the probability that none of them selects his own hat?

We will first use the complementary probability that at least one man selects his own hat. Let E_i be the probability that the ith man selects his own hat. For that, we need to compute $P(E_1 \cup E_2 \cup E_3)$.

Example 1.8 (cont.)

Note that,

$$
\begin{aligned}
P(E_i) &= \frac{1}{3}, \quad i = 1, 2, 3 \\
P(E_i E_j) &= \frac{1}{6}, \quad i \neq j \\
P(E_1 E_2 E_3) &= \frac{1}{6}
\end{aligned}
$$

Why is this correct?

Example 1.8 (cont.)

Now,

$$
\begin{aligned}
P(E_1 \cup E_2 \cup E_3) &= P(E_1) + P(E_2) + P(E_3) - \ldots + P(E_1 E_2 E_3) \\
&= 1 - \frac{1}{2} + \frac{1}{6} \\
&= \frac{2}{3} \Rightarrow 1 - P(E_1 \cup E_2 \cup E_3) = \frac{1}{3}
\end{aligned}
$$

Independent Events

Events E and F are *independent* if

$$P(EF) = P(E)P(F)$$

which also implies

$$P(E|F) = P(E) \text{ and } P(F|E) = P(F)$$

If E and F are not independent, they are said to be *dependent*.

Example 1.9

We toss 2 fair dice. Let E and F be the events that the sum of dice is 6 and the first die equals 4, respectively. Are they independent? Why?

Since we have

$$P(EF) = P(\{4,2\}) = \frac{1}{36} \neq P(E)P(F) = \frac{5}{36}\frac{1}{6} = \frac{5}{216}$$

the events are not independent. How do you interpret this?

Example 1.9 (cont.)

How about if change the definition of event E and let it be the event that the sum of dice is 7. Are they independent now? Why?

Now, since

$$P(EF) = P(\{4,3\}) = \frac{1}{36} = P(E)P(F) = \frac{1}{6}\frac{1}{6} = \frac{1}{36}$$

the events are now independent. Why?

Example 1.10:

Let a ball drawn from an urn containing 4 balls numbered from 1 to 4, and define $E = \{1,2\}$, $F = \{1,3\}$ and $G = \{1,4\}$.

$$\begin{aligned} P(EF) &= P(E)P(F) = \frac{1}{4} \\ P(EG) &= P(E)P(G) = \frac{1}{4} \\ P(FG) &= P(F)P(G) = \frac{1}{4} \end{aligned}$$

However, the events are pairwise independent (but not independent) since

$$\frac{1}{4} = P(EFG) \neq P(E)P(F)P(G) = \frac{1}{8}$$

Independent Trials

Suppose that a sequence of experiments, each of which results either a "success" or a "failure", is to be performed. We let E_i, $i \geq 1$, denote the event that the ith experiment results in a success. If, for all $i_1, i_2, \ldots, i_n$,

$$P(E_{i_1} E_{i_2} \ldots E_{i_n}) = \prod_{j=1}^{n} P(E_{i_j})$$

we say that the sequence of events consists of *independent trials*.

Bayes' Formula

We can write, for events E and F, that $E = EF \cup EF^c$. Since EF and EF^c are mutually exclusive, we can have

$$\begin{aligned} P(E) &= P(EF) + P(EF^c) \\ &= P(E|F)P(F) + P(E|F^c)P(F^c) \\ &= P(E|F)P(F) + P(E|F^c)[1 - P(F)] \end{aligned}$$

How do we interpret this?

Example 1.12

Consider 2 urns. The first one contains 2 white and 7 black balls, and the second one contains 5 white and 6 black balls. We flip a fair coin and draw a ball from the first or second urn depending on whether the outcome was heads or tails. What is the conditional probability that the outcome of the toss was heads given that a white ball was selected?

Let W and H be the events that a white ball is drawn and that the coin comes up heads.

Example 1.12 (cont.)

The desired probability $P(H|W)$ is computed as follows:

$$\begin{aligned} P(H|W) &= \frac{P(HW)}{P(W)} \\ &= \frac{P(W|H)P(H)}{P(W)} \\ &= \frac{P(W|H)P(H)}{P(W|H)P(H) + P(W|H^c)P(H^c)} \\ &= \frac{\frac{2}{9}\frac{1}{2}}{\frac{2}{9}\frac{1}{2} + \frac{5}{11}\frac{1}{2}} \\ &= \frac{22}{67} \end{aligned}$$

Example 1.13

In answering a question on a multiple-choice test a student either knows the answer or guesses. Let p be the probability that she knows the answer and $1-p$ the probability that she guesses. Assume that a student who guesses at the answer will be correct with probability $1/m$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that she answered it correctly?

Let C and K be the events that she answers the question correctly and that she knows the answer, respectively.

Example 1.13 (cont.)

The desired probability $P(K|C)$ is computed as follows:

$$\begin{aligned}
P(K|C) &= \frac{P(KC)}{P(C)} \\
&= \frac{P(C|K)P(K)}{P(C)} \\
&= \frac{P(C|K)P(K)}{P(C|K)P(K)+P(C|K^c)P(K^c)} \\
&= \frac{p}{p+\frac{1}{m}(1-p)} \\
&= \frac{mp}{1+(m-1)p}
\end{aligned}$$

Bayes' Formula

Suppose that $F_1, F_2, \ldots, F_n$ are mutually exclusive events such that

$$\bigcup_{i=1}^{n} F_i = S$$

In other words, exactly one of the events $F_1, F_2, \ldots, F_n$ will occur. We can write

$$E = \bigcup_{i=1}^{n} EF_i$$

Bayes' Formula (cont.)

Since EF_i, $i = 1, \ldots, n$, are mutually exclusive, we obtain that

$$\begin{aligned} P(E) &= \sum_{i=1}^{n} P(EF_i) \\ &= \sum_{i=1}^{n} P(E|F_i)P(F_i) \end{aligned}$$

which states that the $P(E)$ is equal to a weighted average of $P(E|F_i)$, each term being weighted by the probability of the event on which it is conditioned.

Bayes' Formula (cont.)

If we suppose that E has occurred and want to determine which one of the F_j also occurred, we can write the following:

$$\begin{aligned} P(F_j|E) &= \frac{P(EF_j)}{P(E)} \\ &= \frac{P(E|F_j)P(F_j)}{\sum_{i=1}^{n} P(E|F_i)P(F_i)} \end{aligned}$$

which is known as the Bayes' formula.

Example 1.15

We know that a certain letter is equally likely to be in any one of the 3 folders. Let α_i be the probability that we will find the letter in folder i if the letter is, in fact, in folder i, $i = 1, 2, 3$. Suppose that we look for folder 1 and do not find the letter. What is the probability that the latter is there?

We let F_i and E be the events that the letter is in folder i and that we cannot find the letter, respectively.

Example 1.15 (cont.)

The desired probability is

$$\begin{aligned} P(F_1|E) &= \frac{P(E|F_1 P(F_1)}{\sum_{i=1}^{3} P(E|F_i)P(F_i)} \\ &= \frac{(1-\alpha_1)\frac{1}{3}}{(1-\alpha_1)\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} \\ &= \frac{1-\alpha_1}{3-\alpha_1} \end{aligned}$$

The End

Questions?