END3033 Operations Research I
Goal Programming

to accompany

Operations Research: Applications and Algorithms

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Introduction

In some situations, a decision maker may face multiple objectives, and there may be no point in an LP’s feasible region satisfying all objectives. In such a case, how can the decision maker choose a satisfactory decision? Goal programming is one technique that can be used in such situations. The following example illustrates the main ideas of goal programming.

Introduction

Example: The Leon Burnit Advertising Agency is trying to determine a TV advertising schedule for Priceler Auto Company. Priceler has three goals:

- Goal 1: Its ads should be seen by at least 40 million high-income men (HIM).
- Goal 2: Its ads should be seen by at least 60 million low-income people (LIP).
- Goal 3: Its ads should be seen by at least 35 million high-income women (HIW).

Introduction

Leon Burnit can purchase two types of ads: those shown during football games and those shown during soap operas. At most, $600,000 can be spent on ads. The advertising costs and potential audiences of a one-minute ad of each type are shown in below table. Leon Burnit must determine how many football ads and soap opera ads to purchase.

Ad	HIM	LIP	HIW	Cost
Football Game	7	10	5	100,000
Soap Opera	3	5	4	60.000

Table: Millions of Viewers

Introduction

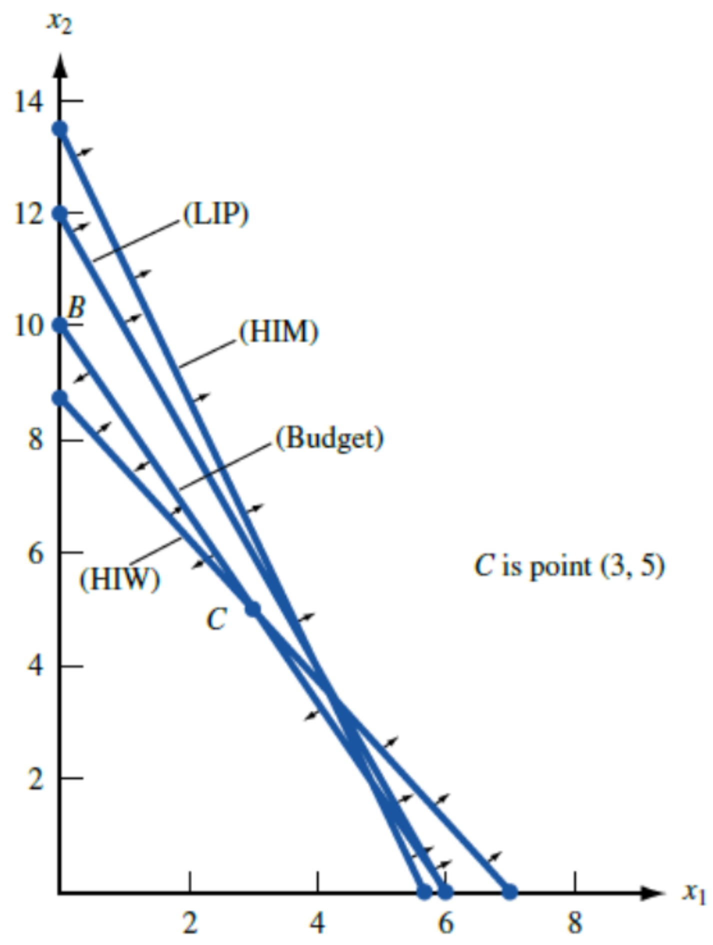
If we let

x_1 = # of minutes of ads shown during football games
x_2 = # of minutes of ads shown during soap operas

We can write the constraints of the problem as

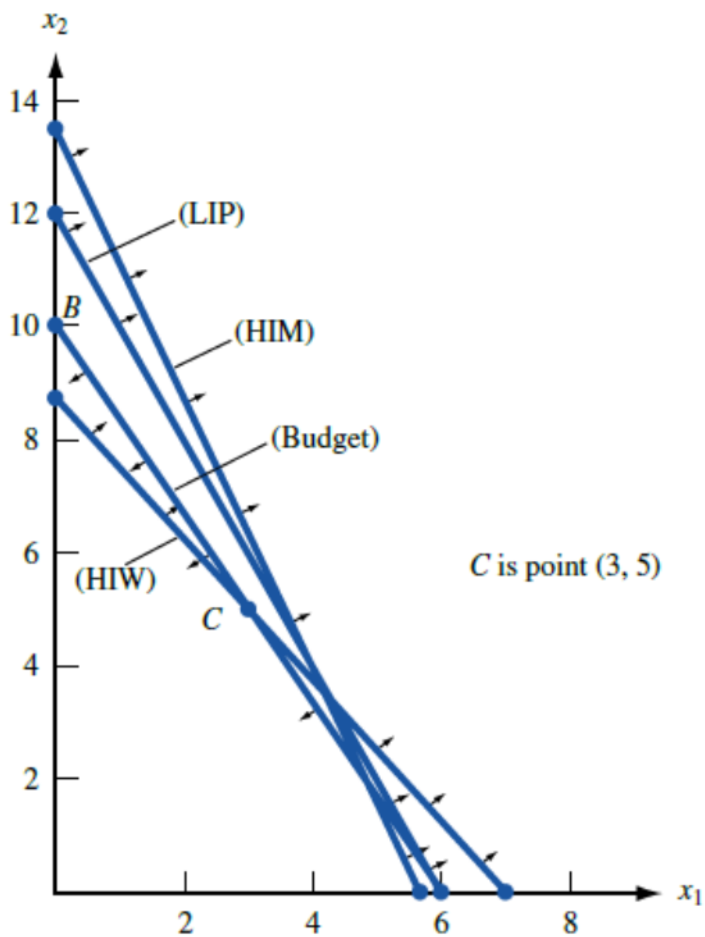
$$\begin{array}{rcrcr} 7x_1 & + & 3x_2 & \geq & 40 \\ 10x_1 & + & 5x_2 & \geq & 60 \\ 5x_1 & + & 4x_2 & \geq & 35 \\ 100x_1 & + & 60x_2 & \leq & 600 \\ x_1 & , & x_2 & \geq & 0 \end{array}$$

Introduction



From the figure, we find that no point that satisfies the budget constraint meets all three of Priceler's goals. Thus, the problem has no feasible solution. It is impossible to meet all of Priceler's goals, so Burnit might ask Priceler to identify, for each goal, a cost (per-unit short of meeting each goal) that is incurred for failing to meet the goal.

Introduction



Burnit can now formulate an LP that minimizes the cost incurred in deviating from Priceler's three goals. The trick is to transform each inequality constraint in that represents one of Priceler's goals into an equality constraint. The cost-minimizing solution might under-satisfy or over-satisfy a given goal, so we need to define the following **deviational variables**:

Introduction

d_i^+ = the amount by which we are over the ith goal
d_i^- = the amount by which we are under the ith goal

Using the deviational variables, we can write

$$\begin{aligned} 7x_1 + 3x_2 + d_1^- - d_1^+ &= 40 \\ 10x_1 + 5x_2 + d_2^- - d_2^+ &= 60 \\ 5x_1 + 4x_2 + d_3^- - d_3^+ &= 35 \end{aligned}$$

Non-Preemptive Goal Programming

Now suppose Priceler determines that

- Each million exposures by which Priceler falls short of the HIM goal costs Priceler a $200,000 penalty because of lost sales.
- Each million exposures by which Priceler falls short of the LIP goal costs Priceler a $100,000 penalty because of lost sales.
- Each million exposures by which Priceler falls short of the HIW goal costs Priceler a $50,000 penalty because of lost sales.

Non-Preemptive Goal Programming

To find the best solution satisfying the above equations, we can write the following model with the objective:

$$\min z = 200d_1^- + 100d_2^- + 50d_3^-$$

$$\begin{array}{rcrcrcrcrcrl}
7x_1 & + & 3x_2 & + & d_1^- & - & d_1^+ & & & = & 40 & \\
10x_1 & + & 5x_2 & + & d_2^- & - & d_2^+ & & & = & 60 & \\
5x_1 & + & 4x_2 & + & d_3^- & - & d_3^+ & & & = & 35 & \\
100x_1 & + & 60x_2 & & & & & + & s_4 & = & 600 & \\
x_1 & , & x_2 & , & d_i^- & , & d_i^+ & , & s_4 & \geq & 0 & , \quad \forall i
\end{array}$$

Non-Preemptive Goal Programming

The optimal solution to the above model is

$$z = 250; x_1 = 6, x_2 = 0$$

$$d_1^+ = 2, d_2^+ = 0, d_3^+ = 0$$

$$d_1^- = 0, d_2^- = 0, d_3^- = 5$$

Interpret the result!

Preemptive Goal Programming

In our LP formulation of the Burnit example, we assumed that Priceler could exactly determine the relative importance of the three goals. For instance, Priceler determined that the HIM goal was 2 times as important as the LIP goal, and the LIP goal was 2 times as important as HIW goal. In many situations, however, a decision maker may not be able to determine precisely the relative importance of the goals.

Preemptive Goal Programming

In our LP formulation of the Burnit example, we assumed that Priceler could exactly determine the relative importance of the three goals. For instance, Priceler determined that the HIM goal was 2 times as important as the LIP goal, and the LIP goal was 2 times as important as HIW goal. In many situations, however, a decision maker may not be able to determine precisely the relative importance of the goals.

Preemptive Goal Programming

When this is the case, preemptive goal programming may prove to be a useful tool. To apply preemptive goal programming, the decision maker must rank his or her goals from the most important (goal 1) to least important (goal n). The objective function coefficient for the variable representing goal i will be P_i where we assume that

$$P_1 \gg P_2 \gg \cdots \gg P_n$$

Preemptive Goal Programming

For the example, we can then write

$$\min z = P_1 d_1^- + P_2 d_2^- + P_3 d_3^-$$

$$\begin{array}{rcrcrcrcrcl}
7x_1 & + & 3x_2 & + & d_1^- & - & d_1^+ & & & = & 40 \\
10x_1 & + & 5x_2 & + & d_2^- & - & d_2^+ & & & = & 60 \\
5x_1 & + & 4x_2 & + & d_3^- & - & d_3^+ & & & = & 35 \\
100x_1 & + & 60x_2 & & & & & + & s_4 & = & 600 \\
x_1 & , & x_2 & , & d_i^- & , & d_i^+ & , & s_4 & \geq & 0
\end{array}$$

Preemptive Goal Programming

Preemptive goal programming problems can be solved by an extension of the simplex known as the **goal programming simplex**. To prepare a problem for solution by the goal programming simplex, we must compute n Row 0s (objective rows), with the ith row corresponding to goal i.

Preemptive Goal Programming

We thus have

Row 0 - Objective 1 (goal 1): $z_1 - P_1 d_1^- = 0$
Row 0 - Objective 2 (goal 2): $z_2 - P_2 d_2^- = 0$
Row 0 - Objective 3 (goal 3): $z_3 - P_3 d_3^- = 0$

By organizing these, we have

Row 0 - Objective 1 (goal 1): $z_1 + 7P_1 x_1 + 3P_1 x_2 - P_1 d_1^+ = 40P_1$
Row 0 - Objective 2 (goal 2): $z_2 + 10P_2 x_1 + 5P_2 x_2 - P_2 d_2^+ = 60P_2$
Row 0 - Objective 3 (goal 3): $z_3 + 5P_3 x_1 + 4P_3 x_2 - P_3 d_3^+ = 30P_3$

Preemptive Goal Programming

	z	x_1	x_2	d_1^+	d_2^+	d_3^+	d_1^-	d_2^-	d_3^-	s_4	RHS
z_1	1	$7P_1$	$3P_1$	$-P_1$	0	0	0	0	0	0	$40P_1$
z_2	1	$10P_2$	$5P_2$	0	$-P_2$	0	0	0	0	0	$60P_2$
z_3	1	$5P_3$	$4P_3$	0	0	$-P_3$	0	0	0	0	$35P_3$
d_1^-	0	7	3	−1	0	0	1	0	0	0	40
d_2^-	0	10	5	0	−1	0	0	1	0	0	65
d_3^-	0	5	4	0	0	−1	0	0	1		35
s_4	0	100	60	0	0	0	0	0	0	1	600

Preemptive Goal Programming

	z	x_1	x_2	d_1^+	d_2^+	d_3^+	d_1^-	d_2^-	d_3^-	s_4	RHS
z_1	1	0	0	0	0	0	$-P_1$	0	0	0	0
z_2	1	0	$\frac{5P_2}{7}$	$\frac{10P_2}{7}$	$-P_2$	0	$-\frac{10P_2}{7}$	0	0	0	$\frac{20P_2}{7}$
z_3	1	0	$\frac{13P_3}{7}$	$\frac{5P_3}{7}$	0	$-P_3$	$-\frac{5P_3}{7}$	0	0	0	$\frac{45P_3}{7}$
x_1	0	1	3/7	– 1/7	0	0	1/7	0	0	0	40/7
d_2^-	0	0	5/7	10/7	– 1	0	– 10/7	1	0	0	20/7
d_3^-	0	0	13/7	5/7	0	– 1	– 5/7	0	1		45/7
s_4	0	0	120/7	100/7	0	0	– 100/7	0	0	1	200/7

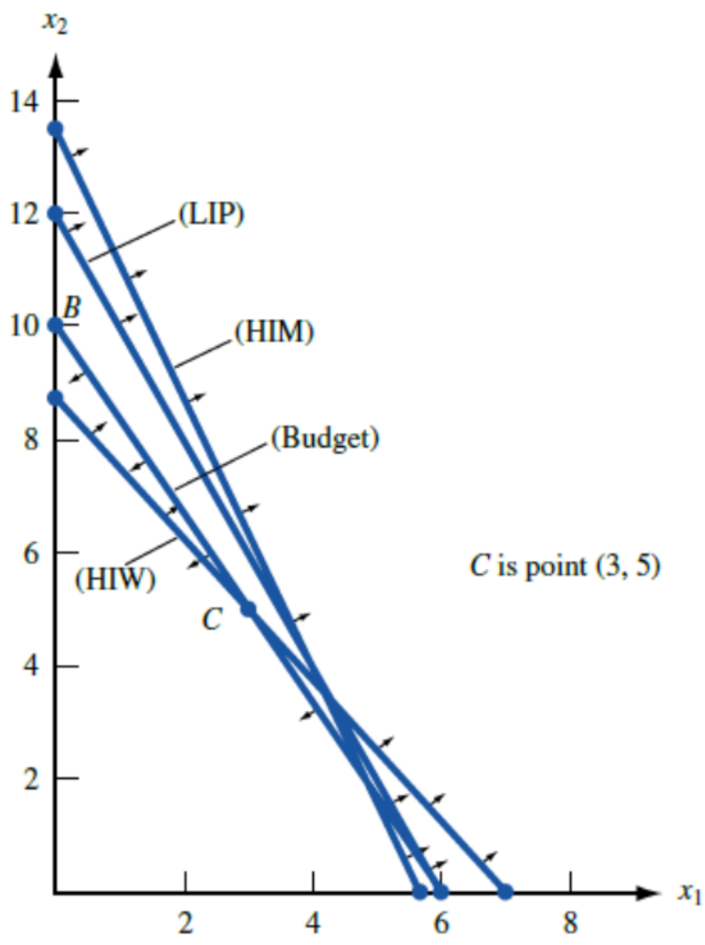
Preemptive Goal Programming

	z	x_1	x_2	d_1^+	d_2^+	d_3^+	d_1^-	d_2^-	d_3^-	s_4	RHS
z_1	1	0	0	0	0	0	$-P_1$	0	0	0	0
z_2	1	0	$-P_2$	0	$-P_2$	0	0	0	0	$-\frac{P_2}{10}$	0
z_3	1	0	P_3	0	0	$-P_3$	0	0	0	$-\frac{P_3}{20}$	$5P_3$
x_1	0	1	3/5	0	0	0	0	0	0	1/100	6
d_2^-	0	0	− 1	0	− 1	0	0	1	0	− 1/10	0
d_3^-	0	0	1	0	0	− 1	0	0	1	− 1/20	5
d_1^+	0	0	6/5	1	0	0	− 1	0	0	7/100	2

Preemptive Goal Programming

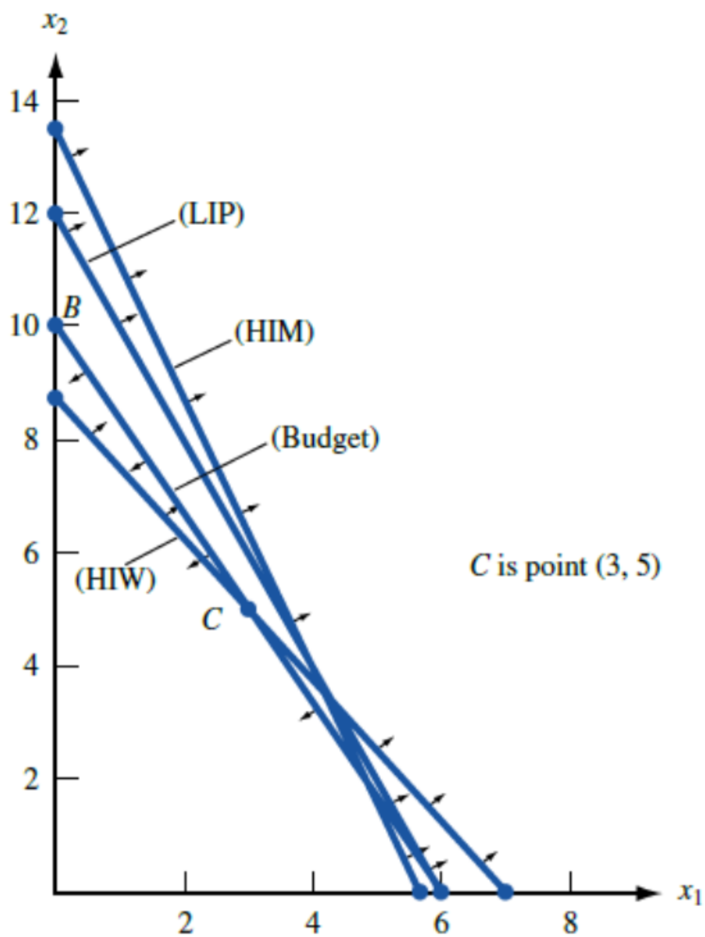
Priorities					Deviational Variables		
Highest	Medium	Lowest	x_1	x_2	HIM	LIP	HIW
HIM	LIP	HIW	6	0	0	0	5
HIM	HIW	LIP	5	5/3	0	5/3	10/3
LIP	HIM	HIW	6	0	0	0	5
LIP	HIW	HIM	6	0	0	0	5
HIW	HIM	LIP	3	5	4	5	0
HIW	LIP	HIM	3	5	4	5	0

Preemptive Goal Programming



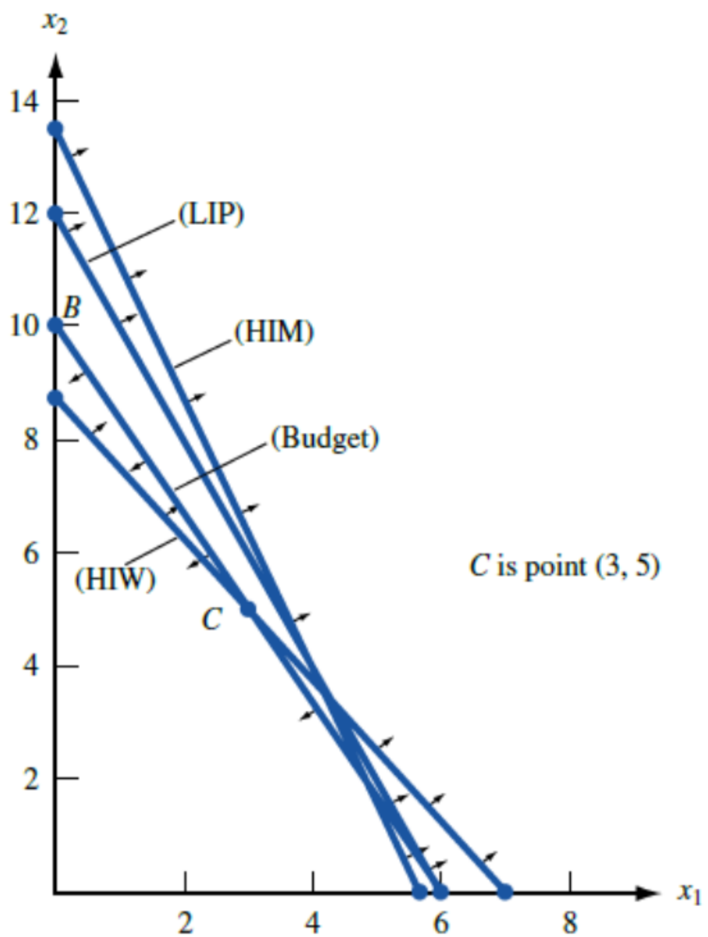
When a preemptive goal programming problem involves only two decision variables, the optimal solution can be found graphically. For example, suppose HIW is the highest priority goal, LIP is the second-highest, and HIM is the lowest. From the Figure, we find that the set of points satisfying the highest-priority goal (HIW) and the budget constraint is bounded by the triangle ABC.

Preemptive Goal Programming



Among these points, we now try to come as close as we can to satisfying the second-highest-priority goal (LIP). Unfortunately, no point in triangle ABC satisfies the LIP goal. We see from the figure, however, that among all points satisfying the highest-priority goal, point C (C is where the HIW goal is exactly met and the budget constraint is binding) is the unique point that comes the closest to satisfying the LIP goal.

Preemptive Goal Programming



Simultaneously solving the following equations, we find that point C (3, 5) is the solution that satisfies both goals and closest to satisfying the LIP goal.

$$
\begin{array}{rcrcr}
5x_1 & + & 4x_2 & = & 35 \\
100x_1 & + & 60x_2 & = & 600
\end{array}
$$

Computer Packages to solve Preemptive GPs

We can use LINDO or MS Excel Solver to solve preemptive GP models. To illustrate how LINDO can be used to solve a preemptive goal programming problem, let's look at the Priceler example with our original set of priorities (HIM followed by LIP followed by HIW).

Thanks… ☺