The Revised Simplex Algorithm

Fatih Cavdur

to accompany Operations Research: Applications and Algorithms by W. Winston

Introduction

- If we know the basic variables, B^{-1}, and the original tableau, we can generate any BFS corresponding to any set of basic variables.
- If we code the simplex algorithm, this is all we need to consider.
- This is the basic idea of the revised simplex algorithm.

Example

Example (Dakota Example):

$$\max z = 60x_1 + 30x_2 + 20x_3$$

s.t.

$$\begin{array}{rcrcrcr}
8x_1 & + & 6x_2 & + & x_3 & \leq & 48 \\
4x_1 & + & 2x_2 & + & 1.5x_3 & \leq & 20 \\
2x_1 & + & 1.5x_2 & + & 0.5x_3 & \leq & 8 \\
 & & x_2 & & & \leq & 5 \\
x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}$$

Example

The IBFS for Dakota:

$$
\begin{array}{rrrrrrrl}
z - & 60x_1 - & 30.0x_2 - & 20.0x_3 & & & & = 0 \\
& 8x_1 + & 6.0x_2 + & x_3 & + s_1 & & & = 24 \\
& 4x_1 + & 2.0x_2 + & 1.5x_3 & & + s_2 & & = 8 \\
& 2x_1 + & 1.5x_2 + & 0.5x_3 & & & + s_3 & = 2
\end{array}
$$

The optimal simplex tableau for Dakota:

$$
\begin{array}{rrrrrrrrl}
z & & + 5.00x_2 & & & & + 10.0s_2 & + 10.0s_3 & = 280 \\
& & - 2.00x_2 & + & + & {\color{red}s_1} & + 2.0s_2 & - 8.0s_3 & = 24 \\
& & - 2.00x_2 & + & {\color{red}x_3} + & & + 2.0s_2 & - 4.0s_3 & = 8 \\
& {\color{red}x_1} & + 1.25x_2 & + & + & & - 0.5s_2 & + 1.5s_3 & = 2
\end{array}
$$

Example: IBFS

For the IBFS, we have

$$\mathbf{x}_B = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \text{ and } \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\mathbf{c}_B = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{c}_N = \begin{bmatrix} 60 & 30 & 20 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: IBFS

We should compute $\bar{c}_j$, $\forall j$, to determine the entering variable.

$$\bar{c}_j = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_j - c_j$$

$$\begin{aligned}
\bar{c}_1 &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_1 - c_1 \\
&= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \\ 2 \end{bmatrix} - 60 \\
&= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \\ 2 \end{bmatrix} - 60 \\
&= -60
\end{aligned}$$

Example: IBFS

Similarly,

$$\begin{aligned}\bar{c}_2 &= \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_2 - c_2 \\ &= [0 \quad 0 \quad 0]\begin{bmatrix}6\\2\\1.5\end{bmatrix} - 30 \\ &= -30\end{aligned}$$

$$\begin{aligned}\bar{c}_3 &= \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_3 - c_3 \\ &= [0 \quad 0 \quad 0]\begin{bmatrix}1\\1.5\\0.5\end{bmatrix} - 20 \\ &= -20\end{aligned}$$

Example: Entering Variable

x_1 enters the basis. To find the leaving variable, we should compute the column of x_1 and the RHS vector.

$$\bar{\mathbf{a}}_1 = \mathbf{B}^{-1}\mathbf{a}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 2 \end{bmatrix}$$

$$\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix}$$

Example: Leaving Variable

By performing the ratio test,

$$\min\begin{Bmatrix} 48/8 \\ 20/4 \\ 8/2 \end{Bmatrix} = 4$$

s_3 leaves the basis and we now have

$$\mathbf{x}_B = \begin{bmatrix} s_1 \\ s_2 \\ x_1 \end{bmatrix} \text{ and } \mathbf{x}_N = \begin{bmatrix} s_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\mathbf{c}_B = [0 \quad 0 \quad 60] \text{ and } \mathbf{c}_N = [0 \quad 30 \quad 20]$$

Example: Next Iteration

The new $\mathbf{B}^{-1}$ can be obtained using the following row operations.

$$R_1 \to R_1 - 4R_3$$
$$R_2 \to R_2 - 2R_3$$
$$R_3 \to \frac{1}{2}R_3$$

As a result, we obtain the following $\mathbf{B}^{-1}$ as follows.

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 0.5 \end{bmatrix}$$

Example: Pricing Out

Now continue with the next iteration:

$$
\begin{aligned}
\bar{c}_2 &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_2 - c_2 \\
&= \begin{bmatrix} 0 & 0 & 60 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1.5 \end{bmatrix} - 30 \\
&= \begin{bmatrix} 0 & 0 & 30 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1.5 \end{bmatrix} - 30 \\
&= 15
\end{aligned}
$$

Example: Pricing Out

Similarly,

$$\begin{aligned}\bar{c}_3 &= \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_3 - c_3 \\ &= [0 \quad 0 \quad 30]\begin{bmatrix}1\\1.5\\0.5\end{bmatrix} - 20 \\ &= -5\end{aligned}$$

$$\begin{aligned}\bar{c}_{s_3} &= \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_{s_3} - c_{s_3} \\ &= [0 \quad 0 \quad 30]\begin{bmatrix}0\\0\\1\end{bmatrix} - 0 \\ &= 30\end{aligned}$$

Example: Entering Variable

x_3 enters the basis. To find the leaving variable, we should compute the column of x_3 and the RHS vector.

$$\bar{\mathbf{a}}_3 = \mathbf{B}^{-1}\mathbf{a}_3 = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} -1 \\ 0.5 \\ 0.25 \end{bmatrix}$$

$$\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ 4 \end{bmatrix}$$

Example: Leaving Variable

By performing the ratio test,

$$\min \begin{Bmatrix} \times \\ 4/0.5 \\ 4/0.25 \end{Bmatrix} = 4$$

s_2 leaves the basis and we now have

$$\mathbf{x}_B = \begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix} \text{ and } \mathbf{x}_N = \begin{bmatrix} s_3 \\ x_2 \\ s_2 \end{bmatrix}$$

$$\mathbf{c}_B = [0 \quad 20 \quad 60] \text{ and } \mathbf{c}_N = [0 \quad 30 \quad 0]$$

Example: Next Iteration

The new $\mathbf{B}^{-1}$ can be obtained using the following row operations.

$$R_1 \to R_1 + 2R_2$$
$$R_2 \to 2R_2$$
$$R_3 \to R_3 - \frac{1}{2}R_2$$

As a result, we obtain the following $\mathbf{B}^{-1}$ as follows.

$$\mathbf{B}^{-1} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix}$$

Example: Pricing Out

Now continue with the next iteration:

$$\begin{aligned}
\bar{c}_2 &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_2 - c_2 \\
&= [0 \quad 20 \quad 60] \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 1.5 \end{bmatrix} - 30 \\
&= [0 \quad 10 \quad 10] \begin{bmatrix} 6 \\ 2 \\ 1.5 \end{bmatrix} - 30 \\
&= 5
\end{aligned}$$

Example: Pricing Out

Similarly,

$$\begin{aligned}
\bar{c}_{s_2} &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_{s_2} - c_{s_2} \\
&= [0 \quad 10 \quad 10] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 0 \\
&= 10
\end{aligned}$$

$$\begin{aligned}
\bar{c}_{s_3} &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_{s_3} - c_{s_3} \\
&= [0 \quad 10 \quad 10] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 0 \\
&= 10
\end{aligned}$$

Example: Optimal Solution

Optimal!

$$\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 24 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix} = \mathbf{x}_B$$

$$z = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 0 & 10 & 10 \end{bmatrix} \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix} = \mathbf{c}_B\bar{\mathbf{b}} = 280$$

Example: Optimal Solution

Once again, the optimal simplex tableau for Dakota:

$$
\begin{array}{rrrrrrrrrrrrl}
z & + & 5.00x_2 & & & & & + & 10.0s_2 & + & 10.0s_3 & = & 280 \\
 & - & 2.00x_2 & + & & + & s_1 & + & 2.0s_2 & - & 8.0s_3 & = & 24 \\
 & - & 2.00x_2 & + & x_3 & + & & + & 2.0s_2 & - & 4.0s_3 & = & 8 \\
x_1 & + & 1.25x_2 & + & & + & & - & 0.5s_2 & + & 1.5s_3 & = & 2
\end{array}
$$

The Product Form of the Inverse

In an iteration, assume that we have found that x_k should enter the basis in row r. We let the column vector for x_k in the current tableau be

$$\bar{\mathbf{a}}_k = \begin{bmatrix} \bar{a}_{1k} \\ \bar{a}_{2k} \\ \dots \\ \bar{a}_{mk} \end{bmatrix}$$

The Product Form of the Inverse

We then define the $m \times m$ matrix $\mathbf{E}$ as

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & \dots & -\dfrac{\bar{a}_{1k}}{\bar{a}_{rk}} & \dots & 0 & 0 \\ 0 & 1 & \dots & -\dfrac{\bar{a}_{2k}}{\bar{a}_{rk}} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -\dfrac{1}{\bar{a}_{rk}} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -\dfrac{\bar{a}_{m-1,k}}{\bar{a}_{rk}} & \dots & 1 & 0 \\ 0 & 0 & \dots & -\dfrac{\bar{a}_{mk}}{\bar{a}_{rk}} & \dots & 0 & 1 \end{bmatrix}$$

The Product Form of the Inverse

Note that $\mathbf{E}$ is simply the $\mathbf{I}_m$ with column r replaced accordingly which is called as an elementary matrix.

The Product Form of the Inverse

We can compute the new row r using the current row r in $\mathbf{B}^{-1}$ as

$$r_n \rightarrow \frac{1}{\bar{a}_{rk}} \times r_c$$

and for $i \neq r$,

$$r_n \rightarrow \frac{\bar{a}_{ik}}{\bar{a}_{rk}} \times r_c$$

The Product Form of the Inverse

We can then write

$$r_n \rightarrow \frac{1}{\bar{a}_{rk}} \times r_c$$

and for $i \neq r$,

$$r_n \rightarrow r_c - \frac{\bar{a}_{ik}}{\bar{a}_{rk}} \times r_c$$

The Product Form of the Inverse

As a result, we can write, by letting $\mathbf{B}_k$ the matrix $\mathbf{B}$ in iteration k,

$$\mathbf{B}_1^{-1} = \mathbf{E}_0\mathbf{B}_0^{-1} = \mathbf{E}_0$$

$$\mathbf{B}_2^{-1} = \mathbf{E}_1\mathbf{B}_1^{-1} = \mathbf{E}_1\mathbf{E}_0$$

and, in general

$$\mathbf{B}_k^{-1} = \mathbf{E}_{k-1}\mathbf{E}_{k-2} \dots \mathbf{E}_1\mathbf{E}_0$$

which is called the product form of the inverse. Most LP software uses the revised simplex method and the product form of the inverse for computational efficiency.

The End

Questions?