END3033 Operations Research I
Sensitivity Analysis & Duality

to accompany

Operations Research: Applications and Algorithms

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Introduction

Consider the following problem where x_1 and x_2 corresponds to the # of product 1 (soldier) and 2 (train) produced per week, and C1, C2 and C3 corresponds to the constraints of resource 1 (finishing), resource 2 (carpentry) and resource 3 (demand), respectively:

$$\max z = 3x_1 + 2x_2$$

s.t.

$$\begin{array}{rcrcrl} 2x_1 & + & x_2 & \leq & 100 & \text{(C1=Finishing)} \\ x_1 & + & x_2 & \leq & 80 & \text{(C2=Carpentry)} \\ x_1 & & & \leq & 40 & \text{(C3=Demand)} \\ x_1 & , & x_2 & \geq & 0 & \end{array}$$

The optimal solution to the problem is $z = 180; x_1 = 20, x_2 = 60$ where $B = \{x_1, x_2, s_3\}$ and $N = \{s_1, s_2\}$.

Introduction

How would the optimal solution to this problem change when we change the parameters of this problem (objective function coefficients, right-hand side terms etc.)? That's sensitivity analysis.

Change in an Objective Function Coefficient

Currently we earn \$3 when we produce a soldier and \$2 when we produce a train. If we increase the profit of producing a soldier sufficiently, would it be still optimal to produce 20 soldier and 60 trains?

We now have

$$x_1 = 20, x_2 = 60 \Rightarrow z = 3x_1 + 2x_2 = 180$$

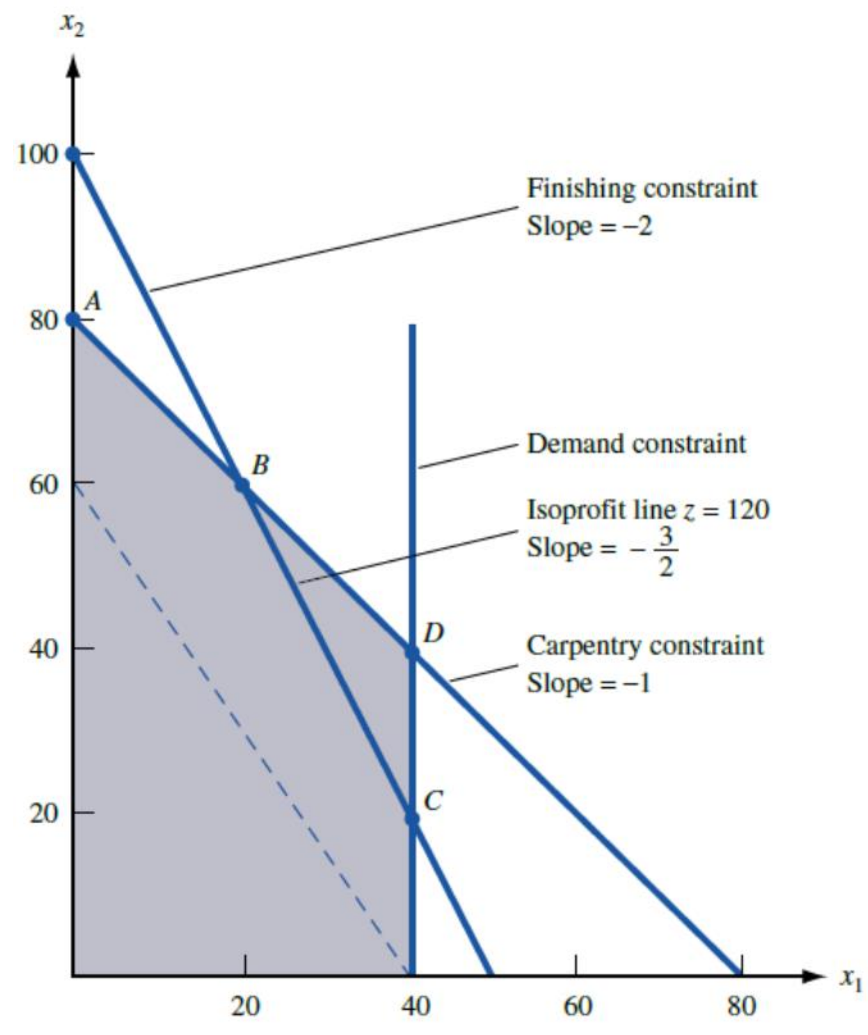
Assume that we fix all other parameters except the coefficient of x_1, and let it be c_1, the contribution to the profit by each soldier. We can then write

$$z = c_1 x_1 + 2x_2 \Rightarrow x_2 = -\frac{c_1}{2} x_1 + \frac{z}{2} \Rightarrow m = -\frac{c_1}{2}$$

Change in an Objective Function Coefficient

We see that if we fix all others and change the coefficient of x_1, we actually change the slope of the function, which means such a change will make the isoprofit line (z-line) flatter or steeper. If we look at the figure, we see that if it is flatter than the carpentry constraint, then, what will be the new optimal point to the problem? Obviously, instead of point B, we will have a new optimal solution at point A! What about if we make the objective function steeper than the finishing constraint?

Change in an Objective Function Coefficient



Change in an Objective Function Coefficient

Now, we can write

$$z = c_1x_1 + 2x_2 \Rightarrow x_2 = -\frac{c_1}{2}x_1 + \frac{z}{2} \Rightarrow m_o = -\frac{c_1}{2}$$
$$2x_1 + x_2 = 100 \Rightarrow x_2 = -2x_1 + 50 \Rightarrow m_{c_1} = -2$$
$$x_1 + x_2 = 80 \Rightarrow x_2 = -1x_1 + 80 \Rightarrow m_{c_2} = -1$$

(1) We can thus write, objective function will be flatter than the carpentry constraint if

$$-\frac{c_1}{2} > -1 \Rightarrow c_1 < 2$$

(2) Similarly, objective function will be steeper than the finishing constraint if

$$-\frac{c_1}{2} < -2 \Rightarrow c_1 > 4$$

Change in an Objective Function Coefficient

Cases (1) and (2) cause the current optimal point change from point B to point A and from point B to point C, respectively, where we will have a new set of basic and non-basic variables.

In other words, we can say the current optimal solution (point B) will remain optimal if

$$\left.\begin{array}{l} -\dfrac{c_1}{2} \ngtr -1 \Rightarrow c_1 \nless 2 \Rightarrow c_1 \geq 2 \\ -\dfrac{c_1}{2} \nless -2 \Rightarrow c_1 \ngtr 4 \Rightarrow c_1 \leq 4 \end{array}\right\} \Rightarrow 2 \leq c_1 \leq 4$$

Change in Right-Hand Side Term

Currently 100 units of resource 1 and 80 units of resource 2 and 40 units of resource 3 are available, respectively. Can we increase or decrease the # of available units for these constrains? Will it change the optimal solution? In other words, for what values of a right-hand side, will the current solution be optimal?

Consider the right-hand side of the first constraint, and let it be b_1. Note that we fix all the other parameters.

Change in Right-Hand Side Term

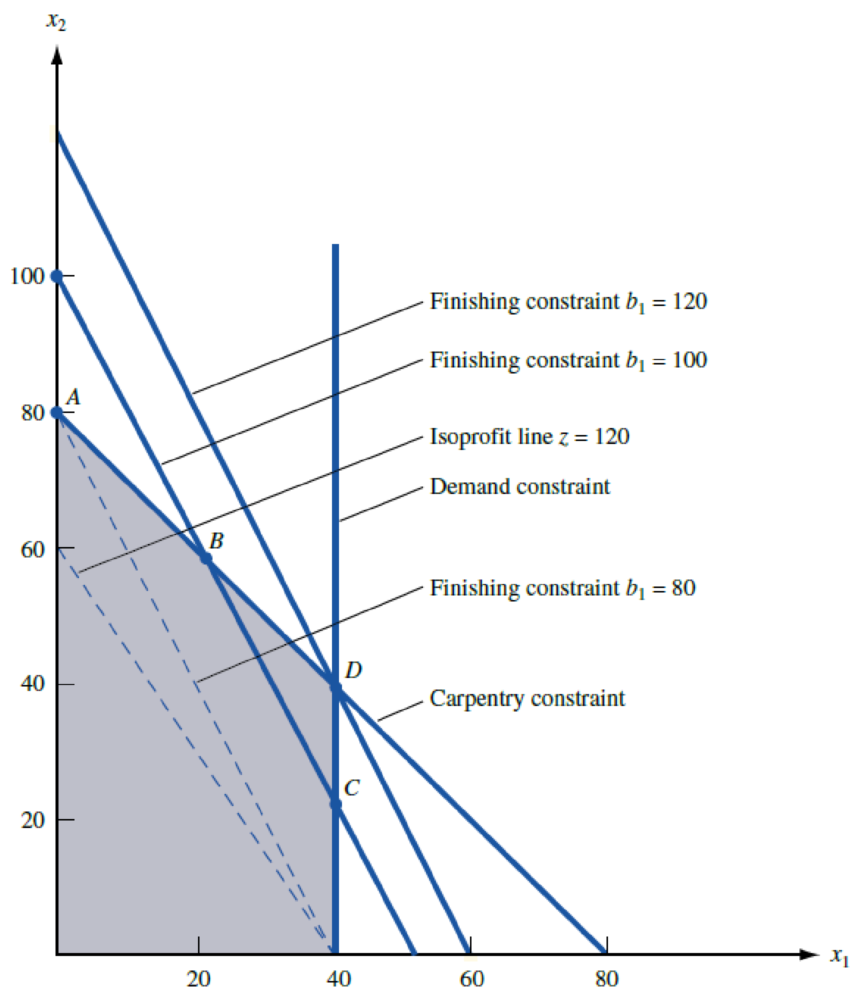
Currently, what are the constraints binding at point B (the optimal solution)? They are the first 2 constraints, finishing and carpentry constraints.

What happens if we change b_1 (currently, it is 100)? These changes will shift the finishing constraint parallel to its current position.

If you are asked to solve this problem graphically, how would you determine the coordinates of the optimal point B?

We note that the optimal point occurs where the first 2 constraints intersect. How will changing one of these right-hand sides change the optimal solution?

Change in Right-Hand Side Term



Change in Right-Hand Side Term

We can thus state the following rule:

As long as the intersection of the first 2 constraints is feasible, current optimal solution will remain optimal.

We see that if we increase b_1 until $b_1 > 120$, where does the intersection of the first 2 constraints occur? What about if we decrase it until $b_1 < 80$?

Note that the current optimal solution remains optimal if

$$80 \leq b_1 \leq 120$$

Shadow Prices

When we change a parameter such as an objective coefficient or right-hand side what changes in a model even if the current optimal solution remains optimal?

Consider the right-hand side of the first constraint, and let Δ be the amount of change on the right-hand side of the first constraint (b_1). We then have

$$\left.\begin{aligned} 2x_1 + x_2 &= 100 + \Delta \\ x_1 + x_2 &= 80 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x_1 &= 20 + \Delta \\ x_2 &= 60 - \Delta \end{aligned}\right\} \Rightarrow z = 3x_1 + 2x_2 = 180 + \Delta$$

What does it mean?

Shadow Prices

We define the shadow price of a constraint as the amount by which the objective value is improved (increase in a max, decrease in a min) when the right-hand side of the constraint is increased by 1 unit.

What is the shadow price of the first constraint in the example? What about the second? What is different about the third constraint?

Shadow Prices

In general, we can write, for constraint i, for a max problem,

$$\begin{pmatrix} \text{new} \\ \text{optimal} \\ z \end{pmatrix} = \begin{pmatrix} \text{old} \\ \text{optimal} \\ z \end{pmatrix} + \begin{pmatrix} \text{shadow} \\ \text{price} \\ i \end{pmatrix} \times (\Delta_i)$$

and for a min problem,

$$\begin{pmatrix} \text{new} \\ \text{optimal} \\ z \end{pmatrix} = \begin{pmatrix} \text{old} \\ \text{optimal} \\ z \end{pmatrix} - \begin{pmatrix} \text{shadow} \\ \text{price} \\ i \end{pmatrix} \times (\Delta_i)$$

Some Definitions

Assume that we have a max LP with n variables and m constraints as follows where we want $\max z$ or $\min z$ such that

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & \ddots & & \vdots & = & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \\
x_1 & , & x_2 & , & \cdots & , & x_n & \geq & 0
\end{array}$$

Some Definitions

For instance, consider the Dakota example without $x_2 \leq 5$ constraint:

$$\max z = 60x_1 + 30x_2 + 20x_3 + 0s_1 + 0s_2 + 0s_3$$

$$\begin{array}{ccccccccccccc}
8x_1 & + & 6.0x_2 & + & x_3 & + & s_1 & & & & & = & 48 \\
4x_1 & + & 2.0x_2 & + & 1.5x_3 & + & & + & s_2 & & & = & 20 \\
2x_1 & + & 1.5x_2 & + & 0.5x_3 & + & & & & + & s_3 & = & 8 \\
x_1 & , & x_2 & , & x_3 & , & s_1 & , & s_2 & , & s_3 & \geq & 0
\end{array}$$

Some Definitions

The optimal solution for Dakota:

$$
\begin{array}{rrrrrrrrrrrrrr}
z & & + & 5.00x_2 & & & & & + & 10s_2 & + & 10.0s_3 & = & 280 \\
 & & - & 2.00x_2 & + & & + & s_1 & + & 2s_2 & - & 8.0s_3 & = & 48 \\
 & & - & 2.00x_2 & + & x_3 & + & & + & 2s_2 & - & 4.0s_3 & = & 20 \\
 & x_1 & + & 1.25x_2 & + & & + & & - & 0.5s_2 & + & 1.5s_3 & = & 8 \\
 & x_1 & , & x_2 & , & x_3 & , & s_1 & , & s_2 & , & s_3 & \geq & 0
\end{array}
$$

Some Definitions

For such an LP with an optimal solution, we define B and N as the set of basic and non-basic variables, and let $\mathbf{x}_B$ and $\mathbf{x}_N$ as the $m \times 1$ and $(n-m) \times 1$ vectors of basic and non-basic variables, respectively.

For this solution, we have

$$\mathbf{x}_B = \begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix} \text{ and } \mathbf{x}_N = \begin{bmatrix} x_2 \\ s_2 \\ s_3 \end{bmatrix}$$

Some Definitions

We define $\mathbf{c}_B$ and $\mathbf{c}_N$ as the $1 \times m$ and $1 \times (n - m)$ vectors objective function coefficients of basic and non-basic variables, respectively.

For our example, we have

$$\mathbf{c}_B = [0 \quad 20 \quad 60] \text{ and } \mathbf{c}_N = [30 \quad 0 \quad 0]$$

Some Definitions

Let $\mathbf{B}$ be the $m \times m$ matrix of constraint coefficients of basic variables in initial BFS.

For our example,

$$\mathbf{B} = \begin{bmatrix} 1 & 1.0 & 8 \\ 0 & 0.5 & 4 \\ 0 & 0.5 & 2 \end{bmatrix}$$

Let $\mathbf{N}$ be the $m \times (n - m)$ matrix of constraint coefficients of non-basic variables in initial BFS.

$$\mathbf{N} = \begin{bmatrix} 6.0 & 0 & 0 \\ 2.0 & 1 & 0 \\ 1.5 & 0 & 1 \end{bmatrix}$$

Some Definitions

Let $\mathbf{a}_j$ be the $m \times 1$ vector of constraint coefficients of variable x_j in initial BFS.

$$\mathbf{a}_2 = \begin{bmatrix} 6.0 \\ 2.0 \\ 1.5 \end{bmatrix} \text{ and } \mathbf{a}_{s_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We define $\mathbf{b}$ as the $m \times 1$ vector of right-hand side terms.

$$\mathbf{b} = \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix}$$

Some Definitions

Our model can be written as

$$z = \mathbf{c}_B \mathbf{x}_B + \mathbf{c}_N \mathbf{x}_N$$

s.t.

$$\begin{array}{rcrcl} \mathbf{B}\mathbf{x}_B & + & \mathbf{N}\mathbf{x}_N & = & \mathbf{b} \\ \mathbf{x}_B & , & \mathbf{x}_N & \geq & \mathbf{0} \end{array}$$

Some Definitions

Using this representation, we can write, for the Dakota example

$$
\begin{aligned}
z &= \begin{bmatrix} 0 & 20 & 60 \end{bmatrix} \begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix} + \begin{bmatrix} 30 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ s_2 \\ s_3 \end{bmatrix} \\
& \begin{bmatrix} 1 & 1.0 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{bmatrix} \begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix} + \begin{bmatrix} 6.0 & 0 & 0 \\ 2.0 & 1 & 0 \\ 1.5 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ s_2 \\ s_3 \end{bmatrix} = 48 \\
& \begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix} , \quad \begin{bmatrix} x_2 \\ s_2 \\ s_3 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

Some Definitions

In the following equations,

$$\mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}$$

If we multiply the expression by $\mathbf{B}^{-1}$, we obtain

$$\mathbf{B}^{-1}\mathbf{B}\mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{B}^{-1}\mathbf{b} \Rightarrow \mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N = \mathbf{B}^{-1}\mathbf{b}$$

Some Definitions

For our Example, we have

$$\mathbf{B} = \begin{bmatrix} 1 & 1.0 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{bmatrix} \Rightarrow \mathbf{B}^{-1} = \begin{bmatrix} 1 & 2.0 & -8.0 \\ 0 & 2.0 & -4.0 \\ 0 & -0.5 & 1.5 \end{bmatrix}$$

We can then write

$$\begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 2.0 & -8.0 \\ 0 & 2.0 & -4.0 \\ 0 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 6.0 & 0 & 0 \\ 2.0 & 1 & 0 \\ 1.5 & 0 & 1 \end{bmatrix}} \begin{bmatrix} x_2 \\ s_2 \\ s_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2.0 & -8.0 \\ 0 & 2.0 & -4.0 \\ 0 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix}}$$

$$\begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix} + \begin{bmatrix} 6.0 & 0 & 0 \\ 2.0 & 1 & 0 \\ 1.5 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 24 \\ 8 \\ 2 \end{bmatrix}$$

Some Definitions

For the objective function, we can write,

$$z = \mathbf{c}_B \mathbf{x}_B + \mathbf{c}_N \mathbf{x}_N \Rightarrow z - \mathbf{c}_B \mathbf{x}_B - \mathbf{c}_N \mathbf{x}_N = 0$$

$$\mathbf{c}_B \mathbf{B}^{-1} \mathbf{B} \mathbf{x}_B + \mathbf{c}_B \mathbf{B}^{-1} \mathbf{N} \mathbf{x}_N = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{b} \Rightarrow \mathbf{c}_B \mathbf{x}_B + \mathbf{c}_B \mathbf{B}^{-1} \mathbf{N} \mathbf{x}_N = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{b}$$

By organizing the above expressions,

$$\begin{array}{ccccccc} z & - & \mathbf{c}_B \mathbf{x}_B & - & \mathbf{c}_N \mathbf{x}_N & = & 0 \\ & & \mathbf{c}_B \mathbf{x}_B & + & \mathbf{c}_B \mathbf{B}^{-1} \mathbf{N} \mathbf{x}_N & = & \mathbf{c}_B \mathbf{B}^{-1} \mathbf{b} \end{array}$$

We obtain

$$z + (\mathbf{c}_B \mathbf{B}^{-1} \mathbf{N} - \mathbf{c}_N) \mathbf{x}_N = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{b}$$

By letting $\bar{c}_j$ be the coefficient of x_j in the objective row,

$$\bar{c}_j = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_j - c_j$$

Some Definitions

For our example, we have

$$\mathbf{c}_B = [0 \quad 20 \quad 60] \text{ and } \mathbf{B}^{-1} = \begin{bmatrix} 1 & 2.0 & -8.0 \\ 0 & 2.0 & -4.0 \\ 0 & -0.5 & 1.5 \end{bmatrix}$$

We can then write

$$\begin{aligned} \bar{c}_2 &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_2 - c_2 \\ &= [0 \quad 20 \quad 60] \begin{bmatrix} 1 & 2.0 & -8.0 \\ 0 & 2.0 & -4.0 \\ 0 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 6.0 \\ 2.0 \\ 1.5 \end{bmatrix} - 30 \\ &= 5 \end{aligned}$$

Some Definitions

$$\begin{aligned}\bar{c}_{s_2} &= \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_{s_2} - c_{s_2}\\ &= \begin{bmatrix}0 & 20 & 60\end{bmatrix}\begin{bmatrix}1 & 2.0 & -8.0\\ 0 & 2.0 & -4.0\\ 0 & -0.5 & 1.5\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} - 0\\ &= 10\end{aligned}$$

$$\begin{aligned}\bar{c}_{s_3} &= \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_{s_3} - c_{s_3}\\ &= \begin{bmatrix}0 & 20 & 60\end{bmatrix}\begin{bmatrix}1 & 2.0 & -8.0\\ 0 & 2.0 & -4.0\\ 0 & -0.5 & 1.5\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} - 0\\ &= 10\end{aligned}$$

Some Definitions

The optimal objective function value is

$$z = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{b} = \begin{bmatrix} 0 & 20 & 60 \end{bmatrix} \begin{bmatrix} 1 & 2.0 & -8.0 \\ 0 & 2.0 & -4.0 \\ 0 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 48 \\ 20 \\ 8 \end{bmatrix} = 280$$

Some Definitions

Example:

Compute the solution for the following LP when $B = \{x_2, s_2\}$.

$$\max z = x_1 + 4x_2$$

$$\begin{array}{rcrcl} x_1 & + & 2x_2 & \leq & 6 \\ 2x_1 & + & x_2 & \leq & 8 \\ x_1 & , & x_2 & \geq & 0 \end{array}$$

Standard form of the model is

$$\max z = x_1 + 4x_2$$

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & s_1 & & & = & 6 \\ 2x_1 & + & x_2 & & & + & s_2 & = & 8 \\ x_1 & , & x_2 & , & s_1 & , & s_2 & \geq & 0 \end{array}$$

Some Definitions

Example:

We start with

$$\mathbf{B} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \mathbf{B}^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix}$$

$$\bar{\mathbf{a}}_1 = \mathbf{B}^{-1}\mathbf{a}_1 = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$$

$$\bar{\mathbf{a}}_{s_1} = \mathbf{B}^{-1}\mathbf{a}_{s_1} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$$

The right-hand side $\bar{\mathbf{b}} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$\bar{c}_1 = \mathbf{c}_B \mathbf{B}^{-1}\mathbf{a}_1 - c_1 = 1$$

$$\bar{c}_{s_1} = \mathbf{c}_B \mathbf{B}^{-1}\mathbf{a}_{s_1} - c_{s_1} = 2$$

Finally, the objective function value is $z = \mathbf{c}_B \mathbf{B}^{-1}\mathbf{b} = 12$

Sensitivity Analysis

- Changing the objective function coefficient of a non-basic variable
- Changing the objective function coefficient of a basic variable
- Changing the right-hand side of an equation
- Changing the column of a non-basic variable
- Adding a new variable (or activity)
- Adding a new constraint

Sensitivity Analysis

In this section, we will consider the following example:

$$\max z = 60x_1 + 30x_2 + 20x_3 + 0s_1 + 0s_2 + 0s_3$$

$$\begin{array}{rcrcrcrcrcrcl}
8x_1 & + & 6.0x_2 & + & x_3 & + & s_1 & & & & & = & 48 \\
4x_1 & + & 2.0x_2 & + & 1.5x_3 & + & & + & s_2 & & & = & 20 \\
2x_1 & + & 1.5x_2 & + & 0.5x_3 & + & & & & + & s_3 & = & 8 \\
x_1 & , & x_2 & , & x_3 & , & s_1 & , & s_2 & , & s_3 & \geq & 0
\end{array}$$

The optimal solution for Dakota:

$$\begin{array}{rcrcrcrcrcrcr}
z & + & 5.00x_2 & & & & & + & 10s_2 & + & 10.0s_3 & = & 280 \\
 & - & 2.00x_2 & + & & + & s_1 & + & 2s_2 & - & 8.0s_3 & = & 48 \\
 & - & 2.00x_2 & + & x_3 & + & & + & 2s_2 & - & 4.0s_3 & = & 20 \\
x_1 & + & 1.25x_2 & + & & + & & - & 0.5s_2 & + & 1.5s_3 & = & 8 \\
x_1 & , & x_2 & , & x_3 & , & s_1 & , & s_2 & , & s_3 & \geq & 0
\end{array}$$

Sensitivity Analysis

Objective Function Coefficient of a Non-Basic Variable

Currently, we have $c_2 = 30$. For what values of c_2, will the current optimal solution remain optimal?

When we change c_2, what will change in the solution?

We can say that, if, (for a max problem, like this), $\bar{c}_2 \geq 0$, the current solution remains optimal.

Let $c_2 = 30 + \Delta$. Since $\mathbf{c}_B\mathbf{B}^{-1} = [0 \quad 10 \quad 10]$, we have

$$\bar{c}_2 = [0 \quad 10 \quad 10]\begin{bmatrix} 6.0 \\ 2.0 \\ 1.5 \end{bmatrix} - (30 + \Delta) = 5 - \Delta$$

We can then write, the current solution is still optimal if

$$\bar{c}_2 \geq 0 \Rightarrow 5 - \Delta \geq 0 \Rightarrow \Delta \leq 5 \Rightarrow c_2 \leq 35$$

Sensitivity Analysis

Objective Function Coefficient of a Non-Basic Variable

If, for instance, $c_2 = 40$, what will happen? We then have

$$\bar{c}_2 = \begin{bmatrix} 0 & 10 & 10 \end{bmatrix} \begin{bmatrix} 6.0 \\ 2.0 \\ 1.5 \end{bmatrix} - 40 = -5$$

meaning the current solution is not optimal anymore. We now have a sub-optimal solution and perform the simplex as follows:

$$\begin{array}{ccccccccccccc}
z & & - & 5.00x_2 & & & & & + & 10.0s_2 & + & 10.0s_3 & = & 280 \\
 & & - & 2.00x_2 & & & + & s_1 & + & 2.0s_2 & - & 8.0s_3 & = & 48 \\
 & & - & 2.00x_2 & + & x_3 & & & + & 2.0s_2 & - & 4.0s_3 & = & 20 \\
 & x_1 & + & \boxed{1.25x_2} & & & & & - & 0.5s_2 & + & 1.5s_3 & = & 8
\end{array}$$

$$\begin{array}{cccccccccccccc}
z & + & 4.0x_1 & & & & & & & + & 8.0s_2 & + & 16.0s_3 & = & 288.0 \\
 & & 1.6x_1 & & & & & + & s_1 & + & 1.2s_2 & - & 5.6s_3 & = & 27.2 \\
 & & 1.6x_1 & & & + & x_3 & + & & + & 1.2s_2 & - & 1.6s_3 & = & 11.2 \\
 & & 0.8x_1 & + & x_2 & & & & & - & 0.4s_2 & + & 1.2s_3 & = & 1.6
\end{array}$$

Sensitivity Analysis

Objective Function Coefficient of a Basic Variable

Consider the objective function coefficient of x_1. Currently we have $c_1 = 60$.

We let $c_1 = 60 + \Delta$, and then,

$$\begin{aligned}\mathbf{c}_B\mathbf{B}^{-1} &= [0 \quad 20 \quad 60+\Delta]\begin{bmatrix}1 & 2.0 & -8.0\\ 0 & 2.0 & -4.0\\ 0 & -0.5 & 1.5\end{bmatrix}\\ &= [0 \quad 10-0.5\Delta \quad 10+1.5\Delta]\end{aligned}$$

Sensitivity Analysis

Objective Function Coefficient of a Basic Variable

Now, we compute the new objective row

$$\bar{c}_2 = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_2 - c_2$$

$$= [0 \quad 10 - 0.5\Delta \quad 10 + 1.5\Delta] \begin{bmatrix} 6.0 \\ 2.0 \\ 1.5 \end{bmatrix} - 30$$

$$= 5 + 1.25\Delta$$

$$\bar{c}_{s_2} = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_{s_2} - c_{s_2}$$

$$= [0 \quad 10 - 0.5\Delta \quad 10 + 1.5\Delta] \begin{bmatrix} 6.0 \\ 2.0 \\ 1.5 \end{bmatrix} - 30$$

$$= 10 - 0.5\Delta$$

$$\bar{c}_{s_3} = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_{s_3} - c_{s_2}$$

$$= [0 \quad 10 - 0.5\Delta \quad 10 + 1.5\Delta] \begin{bmatrix} 6.0 \\ 2.0 \\ 1.5 \end{bmatrix} - 30$$

$$= 10 + 1.5\Delta$$

Sensitivity Analysis

Objective Function Coefficient of a Basic Variable

We thus have

$$\left.\begin{aligned}5+1.25\Delta &\geq 0\\ 10-0.5\Delta &\geq 0\\ 10+1.5\Delta &\geq 0\end{aligned}\right\} \Rightarrow -4 \leq \Delta \leq 20$$

Sensitivity Analysis

Objective Function Coefficient of a Basic Variable

If, $c_1 = 100$, by proceeding similarly, we obtain the sub-optimal solution as follows:

$$
\begin{array}{rrrrrrrrrrrrr}
z & & + & 55.00x_2 & & & & & - & 10.0s_2 & + & 70.0s_3 & = & 360 \\
 & & - & 2.00x_2 & & & + & s_1 & + & 2.0s_2 & - & 8.0s_3 & = & 48 \\
 & & - & 2.00x_2 & + & x_3 & & & + & \boxed{2.0s_2} & - & 4.0s_3 & = & 20 \\
 & x_1 & + & 1.25x_2 & & & & & - & 0.5s_2 & + & 1.5s_3 & = & 8
\end{array}
$$

and obtain the following optimal solution after another iteration:

$$
\begin{array}{rrrrrrrrrrrrr}
z & & + & 45.00x_2 & + & 5.00x_3 & & & & & + & 50.0s_3 & = & 400 \\
 & & & & - & x_3 & + & s_1 & & & - & 4.0s_3 & = & 16 \\
 & & - & x_2 & + & 0.50x_3 & & & & & - & 2.0s_3 & = & 4 \\
 & x_1 & + & 0.75x_2 & + & 0.25x_3 & & & + & s_2 & + & 0.5s_3 & = & 4
\end{array}
$$

Sensitivity Analysis

Reduced Cost

This analysis allows us to define a concept called reduced cost.

The reduced cost for a non-basic variable is the maximum amount by which the variable's objective function coefficient can be increased before the current basis becomes sub-optimal.

Sensitivity Analysis

Changing the Right-Hand Side of an Equation

We can say that as long as the right-hand side of each constraint in the optimal tableau remains non-negative, the current solution remains feasible and optimal.

Sensitivity Analysis

Changing the Right-Hand Side of an Equation

If for instance, if we change b_2 to $b_2 + \Delta$, we have

$$\bar{\mathbf{b}} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 2.0 & -8.0 \\ 0 & 2.0 & -4.0 \\ 0 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 48 \\ 20 + \Delta \\ 8 \end{bmatrix} = \begin{bmatrix} 24 + 2\Delta \\ 8 + 2\Delta \\ 2 - 0.5\Delta \end{bmatrix}$$

We thus have

$$\left.\begin{aligned} 24 + 2.0\Delta \geq 0 \\ 8 + 2.0\Delta \geq 0 \\ 2 - 0.5\Delta \geq 0 \end{aligned}\right\} \Rightarrow -4 \leq \Delta \leq 4 \Rightarrow 16 \leq b_2 \leq 24$$

Sensitivity Analysis

Changing the Right-Hand Side of an Equation

What about the variables and objective function? If, for instance, $b_2 = 22$, we have

$$\begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 2.0 & -8.0 \\ 0 & 2.0 & -4.0 \\ 0 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 48 \\ 22 \\ 8 \end{bmatrix} = \begin{bmatrix} 28 \\ 12 \\ 1 \end{bmatrix}$$

and

$$z = \mathbf{c}_B \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 0 & 10 & 10 \end{bmatrix} \begin{bmatrix} 48 \\ 22 \\ 8 \end{bmatrix} = 300$$

Sensitivity Analysis

Changing the Right-Hand Side of an Equation

If $b_2 = 30$, current solution is no longer optimal. We then have

$$\begin{bmatrix} s_1 \\ x_3 \\ x_1 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & 2.0 & -8.0 \\ 0 & 2.0 & -4.0 \\ 0 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 48 \\ 30 \\ 8 \end{bmatrix} = \begin{bmatrix} 44 \\ 28 \\ -3 \end{bmatrix}$$

and

$$z = \mathbf{c}_B \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 0 & 10 & 10 \end{bmatrix} \begin{bmatrix} 48 \\ 30 \\ 8 \end{bmatrix} = 380$$

Sensitivity Analysis

Changing the Right-Hand Side of an Equation

$$
\begin{array}{rrrrrrrrrrrrrr}
z & & + & 5.00x_2 & & & & & + & 10.0s_2 & + & 10.0s_3 & = & 480 \\
 & & - & 2.00x_2 & & & + & s_1 & + & 2.0s_2 & - & 8.0s_3 & = & 44 \\
 & & - & 2.00x_2 & + & x_3 & & & + & 2.0s_2 & - & 4.0s_3 & = & 28 \\
 & x_1 & + & 1.25x_2 & & & & & - & 0.5s_2 & + & 1.5s_3 & = & -3
\end{array}
$$

What to do now? We will come back to this later!!!

Sensitivity Analysis

Changing the Column of a Non-Basic Variable

When the column of a non-basic variable changes, we need to check the objective row of the variable if it violates the optimality condition or not. To do it, we price out that variable, that is, we compute,

$$\bar{c}_j = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_j - c_j$$

Sensitivity Analysis

Changing the Column of a Non-Basic Variable

If, for instance, we change the column of the non-basic variable x_2 from $c_2 = 30$ to $c_2 = 43$ and $\mathbf{a}_2 = [6 \quad 2 \quad 1.5]^T$ to $\mathbf{a}_2 = [5 \quad 2 \quad 2]^T$,

$$\begin{aligned}\bar{c}_2 &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_2 - c_2 \\ &= [0 \quad 10 \quad 10] \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix} - 43 \\ &= -3\end{aligned}$$

which means the current solution is no longer optimal. We then have the following sub-optimal solution:

Sensitivity Analysis

Adding a New Variable or Activity

Assume that we have a new product that can be sold for \$15 and use 1 board foot of lumber, and 1 hour of carpentry, finishing hours. That is

$$c_4 = 15 \text{ and } \mathbf{a}_4 = [1 \quad 1 \quad 1]^T$$

To find out if the current solution remains optimal, we price out the new variable.

$$\begin{aligned}
\bar{c}_4 &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_4 - c_4 \\
&= [0 \quad 10 \quad 10] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 15 \\
&= 5 \geq 0 \Rightarrow \text{current solution remains optimal}
\end{aligned}$$

Sensitivity Analysis

Summary

We can summarize our discussion using the following table:

Change	Effect	Optimality Condition
Changing c_j (non-basic)	Compute $\bar{c}_j$	$\bar{c}_j \geq 0$
Changing c_j (basic)	Compute Row 0	$\bar{\mathbf{c}} \geq 0$
Changing b_i	Compute RHS	$\bar{\mathbf{b}} \geq 0$
Changing $\mathbf{a}_j$	Compute $\bar{\mathbf{a}}_j$ and $\bar{c}_j$	$\bar{c}_j \geq 0$
New Variable x_j	Compute $\bar{\mathbf{a}}_j$ and $\bar{c}_j$	$\bar{c}_j \geq 0$

Sensitivity Analysis

Multiple Parameter Changes (100% Rule)-Objective Function

Case 1:

All variables whose objective function coefficients are changed have nonzero reduced costs in the optimal row 0.

Case 2:

At least one variable whose objective function coefficient is changed has a reduced cost of zero.

Sensitivity Analysis

Multiple Parameter Changes (100% Rule)-Objective Function

In Case 1, the current basis remains optimal if and only if the objective function coefficient for each variable remains within the allowable range. If the current basis remains optimal, then both the values of the decision variables and objective function remain unchanged. If the objective function coefficient for any variable is outside its allowable range, then the current basis is no longer optimal.

Sensitivity Analysis

Multiple Parameter Changes (100% Rule)-Objective Function

In Case 2, we can often show that the current basis remains optimal by using the 100% Rule. Let

c_j = original objective function coefficient

Δc_j = change in c_j

i_j = max allowable increase in c_j

d_j = max allowable decrease in c_j

Sensitivity Analysis

Multiple Parameter Changes (100% Rule)-Objective Function

We then define the ratio, r_j as

$$r_j = \begin{cases} \dfrac{\Delta c_j}{i_j}, & \Delta c_j \geq 0 \\ -\dfrac{\Delta c_j}{d_j}, & \Delta c_j \leq 0 \end{cases}$$

$$\sum_j r_j \leq 1 \Rightarrow \text{current solution is optimal}$$

$$\sum_j r_j > 1 \Rightarrow \text{current solution might or might not be optimal}$$

Sensitivity Analysis

Multiple Parameter Changes (100% Rule)-RHS

Case 1:

All constraints whose right-hand sides are being modified are nonbinding constraints.

Case 2:

At least one of the constraints whose right-hand side is being modified is a binding constraint.

Sensitivity Analysis

Multiple Parameter Changes (100% Rule)-RHS

In Case 1, the current basis remains optimal if and only if the objective function coefficient for each variable remains within the allowable range. If the current basis remains optimal, then both the values of the decision variables and objective function remain unchanged. If the objective function coefficient for any variable is outside its allowable range, then the current basis is no longer optimal.

Sensitivity Analysis

Multiple Parameter Changes (100% Rule)-RHS

In Case 2, we can often show that the current basis remains optimal by using the 100% Rule. Let

b_j = original right-hand side term

Δb_j = change in b_j

i_j = max allowable increase in b_j

d_j = max allowable decrease in b_j

Sensitivity Analysis

Multiple Parameter Changes (100% Rule)-RHS

We then define the ratio, r_j as

$$r_j = \begin{cases} \dfrac{\Delta b_j}{i_j}, & \Delta b_j \geq 0 \\ -\dfrac{\Delta b_j}{d_j}, & \Delta b_j \leq 0 \end{cases}$$

$$\sum_j r_j \leq 1 \Rightarrow \text{current solution is optimal}$$

$$\sum_j r_j > 1 \Rightarrow \text{current solution might or might not be optimal}$$

Sensitivity Analysis

Multiple Parameter Changes (100% Rule)

Example for Case 1:

Consider the following example (Diet Problem):

$$\min z = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

$$\begin{array}{rcrcrcrcr}
400x_1 & + & 200x_2 & + & 150x_3 & + & 500x_4 & \geq & 500 \\
3x_1 & + & 2x_2 & + & & + & & \geq & 6 \\
2x_1 & + & 2x_2 & + & 4x_3 & + & 4x_4 & \geq & 10 \\
2x_1 & + & 4x_2 & + & x_3 & + & 5x_4 & \geq & 8 \\
x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}$$

Sensitivity Analysis

Multiple Parameter Changes (100% Rule)

We have the following ranges for the corresponding parameters:

$$c_1 = 50 \rightarrow -27.5 \leq \Delta < \infty$$
$$c_2 = 20 \rightarrow -5 \leq \Delta < 18.333$$
$$c_3 = 30 \rightarrow -30 \leq \Delta < 10$$
$$c_4 = 80 \rightarrow -50 \leq \Delta < \infty$$

If we change the parameters as $c_1 = 60$ and $c_4 = 50$, since both have non-zero reduced costs, we have

$$50 - 27.5 = 22.5 \leq c_1 = 60 \leq 50 + \infty = \infty$$

$$80 - 50 = 30 \leq c_4 = 50 \leq 80 + \infty = \infty$$

Sensitivity Analysis

Multiple Parameter Changes (100% Rule)

Another Example for Case 1:

If we change the parameters as $c_1 = 40$ and $c_4 = 25$, since both have non-zero reduced costs, we have

$$50 - 27.5 = 22.5 \leq c_1 = 40 \leq 50 + \infty = \infty$$

$$80 - 50 = 30 \nleq c_4 = 25 \leq 80 + \infty = \infty$$

The current solution is no longer optimal. The problem must be solved again.

Sensitivity Analysis

Multiple Parameter Changes (100% Rule)

Example for Case 2:

Consider the following example (Dakota Problem):

$$\max z = 60x_1 + 30x_2 + 20x_3 + 0s_1 + 0s_2 + 0s_3$$

$$\begin{array}{ccccccccccccc}
8x_1 & + & 6.0x_2 & + & x_3 & + & s_1 & & & & & = & 48 \\
4x_1 & + & 2.0x_2 & + & 1.5x_3 & + & & + & s_2 & & & = & 20 \\
2x_1 & + & 1.5x_2 & + & 0.5x_3 & + & & & & + & s_3 & = & 8 \\
x_1 & , & x_2 & , & x_3 & , & s_1 & , & s_2 & , & s_3 & \geq & 0
\end{array}$$

Sensitivity Analysis

Multiple Parameter Changes (100% Rule)

The optimal solution for Dakota:

$$
\begin{array}{rcrcrcrcrcrcr}
z & + & 5.00x_2 & & & & & + & 10.0s_2 & + & 10.0s_3 & = & 280 \\
 & - & 2.00x_2 & + & & + & s_1 & + & 2.0s_2 & - & 8.0s_3 & = & 24 \\
 & - & 2.00x_2 & + & x_3 & + & & + & 2.0s_2 & - & 4.0s_3 & = & 8 \\
x_1 & + & 1.25x_2 & + & & + & & - & 0.5s_2 & + & 1.5s_3 & = & 2 \\
x_1 & , & x_2 & , & x_3 & , & s_1 & , & s_2 & , & s_3 & \geq & 0
\end{array}
$$

Sensitivity Analysis

Multiple Parameter Changes (100% Rule)

We have the following ranges for the corresponding parameters:

$$c_1 = 60 \rightarrow -4 \leq \Delta < 20$$
$$c_2 = 30 \rightarrow -\infty \leq \Delta < 5$$
$$c_3 = 20 \rightarrow -5 \leq \Delta < 2.5$$

Sensitivity Analysis

Multiple Parameter Changes (100% Rule)

If we change the parameters as $c_1 = 70$, $c_3 = 18$,

$$r_1 = \frac{\Delta c_1}{i_1} = \frac{10}{20} = 0.5$$

$$r_2 = 0$$

$$r_3 = \frac{\Delta c_3}{i_3} = \frac{2}{5} = 0.4$$

$$\sum_{j=1}^{3} r_j = 0.9 \leq 1 \Rightarrow \text{current solution is optimal}$$

Duality

Consider the following max problem (normal max problem):

$$\max z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & \leq & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & \leq & b_2 \\
\vdots & & \vdots & & \ddots & & \vdots & \leq & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & \leq & b_m \\
x_1 & , & x_2 & , & \cdots & , & x_n & \geq & 0
\end{array}$$

Duality

The dual of the problem is given as follows (normal min problem):

$$\min w = b_1 y_1 + b_2 y_2 + \cdots + b_m y_m$$

$$\begin{array}{rcrcccrcc}
a_{11} y_1 & + & a_{21} y_2 & + & \cdots & + & a_{m1} y_m & \geq & c_1 \\
a_{12} y_1 & + & a_{22} y_2 & + & \cdots & + & a_{m2} y_m & \geq & c_2 \\
\vdots & & \vdots & & \ddots & & \vdots & \geq & \vdots \\
a_{1n} y_1 & + & a_{2n} y_2 & + & \cdots & + & a_{mn} y_m & \geq & c_n \\
y_1 & , & y_2 & , & \cdots & , & y_m & \geq & 0
\end{array}$$

Duality

Consider the Dakota Example:

$$\max z = 60x_1 + 30x_2 + 20x_3$$

$$\begin{array}{rcrcrcr}
8x_1 & + & 6.0x_2 & + & x_3 & \leq & 48 \\
4x_1 & + & 2.0x_2 & + & 1.5x_3 & \leq & 20 \\
2x_1 & + & 1.5x_2 & + & 0.5x_3 & \leq & 8 \\
x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}$$

Duality

The dual of the problem is then

$$\min w = 48y_1 + 20y_2 + 8y_3$$

$$\begin{array}{rcrcrcr}
8y_1 & + & 4.0y_2 & + & 2y_3 & \geq & 60 \\
6y_1 & + & 2.0y_2 & + & 1.5y_3 & \geq & 30 \\
y_1 & + & 1.5y_2 & + & 0.5y_3 & \geq & 20 \\
y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}$$

Duality

Consider the Diet Example:

$$\min w = 50y_1 + 20y_2 + 30y_3 + 80y_4$$

$$\begin{array}{rcrcrcrcr}
400y_1 & + & 200y_2 & + & 150y_3 & + & 500y_4 & \geq & 500 \\
3y_1 & + & 2y_2 & & & & & \geq & 6 \\
2y_1 & + & 2y_2 & + & 4y_3 & + & 4y_4 & \geq & 10 \\
2y_1 & + & 2y_2 & + & y_3 & + & 5y_4 & \geq & 8 \\
y_1 & , & y_2 & , & y_3 & , & y_4 & \geq & 0
\end{array}$$

Duality

The dual of the problem is then

$$\max z = 500x_1 + 6x_2 + 10x_3 + 8x_4$$

$$\begin{array}{rcrcrcrcr}
400x_1 & + & 3x_2 & + & 2x_3 & + & 2x_4 & \leq & 50 \\
200x_1 & + & 2x_2 & + & 2x_3 & + & 4x_4 & \leq & 20 \\
150x_1 & & & + & 4x_3 & + & x_4 & \leq & 30 \\
500x_1 & & & + & 4x_3 & + & 5x_4 & \leq & 80 \\
x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}$$

Duality

If we have a non-normal LP, we can consider the following 2 alternatives:

- Transform the non-normal LP to a normal one and write the dual of the LP.
- Directly write the dual of the non-normal LP.

Duality

Example of a Non-Normal LP:

$$\max z = 2x_1 + x_2$$

$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 2 \\ 2x_1 & - & x_2 & \geq & 3 \\ x_1 & - & x_2 & \leq & 1 \\ x_1 & & & \geq & 0 \\ & & x_2 & : & \text{urs} \end{array}$$

Duality

Another Example of a Non-Normal LP:

$$\min w = 2y_1 + 4y_2 + 6y_3$$

$$\begin{array}{rcrcrcr}
y_1 & + & 2y_2 & + & y_3 & \geq & 2 \\
y_1 & & & - & y_3 & \geq & 1 \\
 & & y_2 & + & y_3 & = & 1 \\
2y_1 & + & y_2 & & & \leq & 3 \\
y_1 & & & & & : & \text{urs} \\
 & & y_2 & , & y_3 & \geq & 0
\end{array}$$

Duality

Transform the non-normal LP to a normal one and write the dual of the LP as

- Convert each $\leq$ $(\geq)$ type constraint to a $\geq$ $(\leq)$ type constraint.
- Convert each equation into two inequalities.
- Represent each unrestricted variable using two non-negative variables.
- Write the dual of the transformed normal LP.

Duality

Directly write the dual of the non-normal max (min) LP as

- For each constraint and variable in normal form, write the corresponding dual variable and constraint as before.
- For each $\geq$ ($\leq$) type constraint, the corresponding dual variable is non-positive.
- For each non-positive variable, the corresponding dual constraint is a $\leq$ ($\geq$) type constraint.

Duality

Consider the Dakota Example:

$$\max z = 60x_1 + 30x_2 + 20x_3$$

$$\begin{array}{rcrcrcr}
8x_1 & + & 6.0x_2 & + & x_3 & \leq & 48 \\
4x_1 & + & 2.0x_2 & + & 1.5x_3 & \leq & 20 \\
2x_1 & + & 1.5x_2 & + & 0.5x_3 & \leq & 8 \\
x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}$$

Duality

The dual of the problem is then

$$\min w = 48y_1 + 20y_2 + 8y_3$$

$$\begin{array}{rcrcrcr}
8y_1 & + & 4.0y_2 & + & 2y_3 & \geq & 60 \\
6y_1 & + & 2.0y_2 & + & 1.5y_3 & \geq & 30 \\
y_1 & + & 1.5y_2 & + & 0.5y_3 & \geq & 20 \\
y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}$$

Duality

Note that the first constraint is associated with desks, the second with tables and the third with chairs. Also, note that y_1, y_2 and y_3 are associated with lumber, finishing hours and carpentry hours, respectively. Suppose that someone wants to purchase all of Dakota's resources. She must determine the price she is willing to pay for a unit of each resource. We thus define

- $y_1 =$ price for 1 board ft. of lumber
- $y_2 =$ price for 1 finishing hour
- $y_3 =$ price for 1 carpentry hour

Duality

Consider the Diet Example:

$$\min w = 50y_1 + 20y_2 + 30y_3 + 80y_4$$

$$\begin{array}{rcrcrcrcr}
400y_1 & + & 200y_2 & + & 150y_3 & + & 500y_4 & \geq & 500 \\
3y_1 & + & 2y_2 & & & & & \geq & 6 \\
2y_1 & + & 2y_2 & + & 4y_3 & + & 4y_4 & \geq & 10 \\
2y_1 & + & 2y_2 & + & y_3 & + & 5y_4 & \geq & 8 \\
y_1 & , & y_2 & , & y_3 & , & y_4 & \geq & 0
\end{array}$$

Duality

The dual of the problem is then

$$\max z = 500x_1 + 6x_2 + 10x_3 + 8x_4$$

$$\begin{array}{rcrcrcrcr}
400x_1 & + & 3x_2 & + & 2x_3 & + & 2x_4 & \leq & 50 \\
200x_1 & + & 2x_2 & + & 2x_3 & + & 4x_4 & \leq & 20 \\
150x_1 & & & + & 4x_3 & + & x_4 & \leq & 30 \\
500x_1 & & & + & 4x_3 & + & 5x_4 & \leq & 80 \\
x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0
\end{array}$$

Duality

Now suppose that there is a salesperson that sells calories, chocolate, sugar and fat and wants to ensure that a dieter will meet all of our daily requirements by purchasing calories, chocolate, sugar and fat while maximizing her profit. We must then determine

- x_1 = price per calorie to charge dieter
- x_2 = price per ounce of chocolate to charge dieter
- x_3 = price per ounce of sugar to charge dieter
- x_4 = price per ounce of fat to charge dieter

Duality

Consider the following max problem (normal max problem):

$$\max z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & \leq & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & \leq & b_2 \\
\vdots & & \vdots & & \ddots & & \vdots & \leq & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & \leq & b_m \\
x_1 & , & x_2 & , & \cdots & , & x_n & \geq & 0
\end{array}$$

Duality

The dual of the problem is given as follows (normal min problem):

$$\min w = b_1 y_1 + b_2 y_2 + \cdots + b_m y_m$$

$$\begin{array}{ccccccccc}
a_{11} y_1 & + & a_{21} y_2 & + & \cdots & + & a_{m1} y_m & \geq & c_1 \\
a_{12} y_1 & + & a_{22} y_2 & + & \cdots & + & a_{m2} y_m & \geq & c_2 \\
\vdots & & \vdots & & \ddots & & \vdots & \geq & \vdots \\
a_{1n} y_1 & + & a_{2n} y_2 & + & \cdots & + & a_{mn} y_m & \geq & c_n \\
y_1 & , & y_2 & , & \cdots & , & y_m & \geq & 0
\end{array}$$

Duality

Lemma (Weak Duality):

Let $\mathbf{x}^T = [x_1 \quad x_2 \quad \cdots \quad x_n]$ and $\mathbf{y} = [y_1 \quad y_2 \quad \cdots \quad y_m]$ be any feasible solutions to the primal and dual, respectively. We can then write

$$z \leq w$$

Duality

Proof:

Multiply the ith primal constraint with $y_i \geq 0$,

$$\sum_{j=1}^{n} y_i a_{ij} x_j \leq b_i y_i, \quad i = 1, \ldots, m$$

If we sum the above expressions for all primal constraints,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} y_i a_{ij} x_j \leq \sum_{i=1}^{m} b_i y_i$$

Duality

Similarly, multiply the jth dual constraint with $x_j \geq 0$,

$$\sum_{i=1}^{m} x_j a_{ij} y_j \geq c_j x_j, \quad j = 1, \ldots, n$$

If we sum the above expressions for all dual constraints,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} y_i a_{ij} x_j \geq \sum_{j=1}^{n} c_j x_j$$

Duality

Combining the above expressions, we obtain

$$\underbrace{\sum_{j=1}^{n} c_j x_j}_{z} \leq \sum_{i=1}^{m} \sum_{j=1}^{n} y_i a_{ij} x_j \leq \underbrace{\sum_{i=1}^{m} b_i y_i}_{w}$$

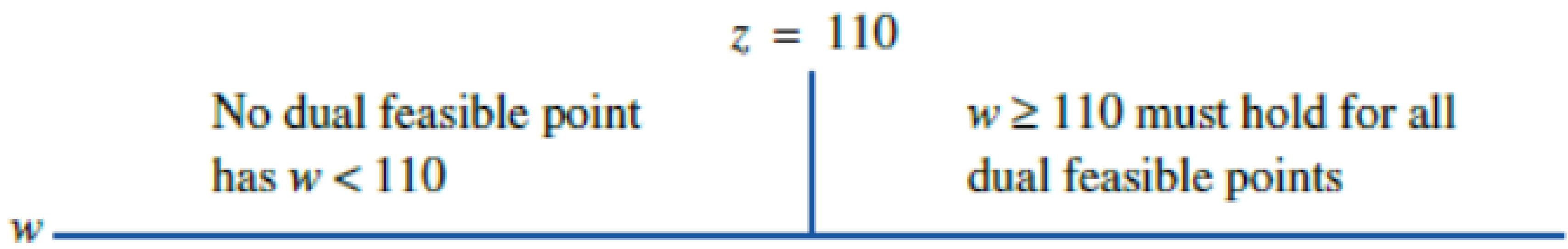
Duality

For the Dakota Example, we see that $(x_1, x_2, x_3) = (1,1,1)$ is a primal feasible solution with $z = 110$. Weak Duality implies that any dual feasible solution (y_1, y_2, y_3) must satisfy

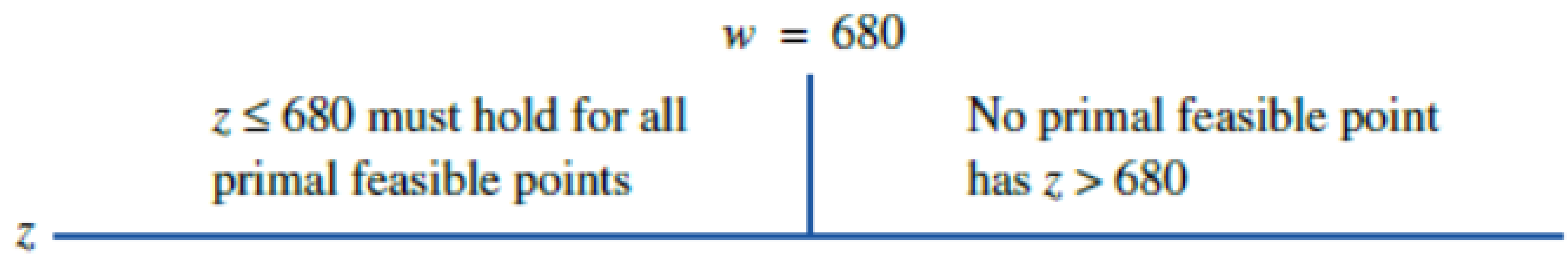
$$48y_1 + 20y_2 + 8y_3 \geq 110$$

Check that!

Duality



$z = 110$
No dual feasible point has $w < 110$
$w \geq 110$ must hold for all dual feasible points
w



$w = 680$
$z \leq 680$ must hold for all primal feasible points
No primal feasible point has $z > 680$
z

Duality

Lemma:

Let $\bar{\mathbf{x}}^T = [x_1 \quad x_2 \quad \cdots \quad x_n]$ and $\bar{\mathbf{y}} = [y_1 \quad y_2 \quad \cdots \quad y_m]$ be any feasible solutions to the primal and dual, respectively.

If $\mathbf{c}\bar{\mathbf{x}} = \bar{\mathbf{y}}\mathbf{b}$, $\bar{\mathbf{x}}$ is optimal for the primal and $\bar{\mathbf{y}}$ is optimal for the dual.

Duality

Lemma:

If the primal is unbounded, then, the dual is infeasible.

Lemma:

If the dual is unbounded, then, the primal is infeasible.

Duality

Theorem:

If B is the set of an optimal basis for primal, then, $c_B B^{-1}$ is an optimal solution to the dual and $\bar{z} = \overline{w}$.

Duality

The optimal solution for Dakota:

$$
\begin{array}{rrrrrrrrrrrrrr}
z & + & 5.00x_2 & & & & 0s_1 & + & 10.0s_2 & + & 10.0s_3 & = & 280 \\
 & - & 2.00x_2 & + & & + & s_1 & + & 2.0s_2 & - & 8.0s_3 & = & 24 \\
 & - & 2.00x_2 & + & x_3 & + & & + & 2.0s_2 & - & 4.0s_3 & = & 8 \\
x_1 & + & 1.25x_2 & + & & + & & - & 0.5s_2 & + & 1.5s_3 & = & 2 \\
x_1 & , & x_2 & , & x_3 & , & s_1 & , & s_2 & , & s_3 & \geq & 0
\end{array}
$$

Duality

$$
\begin{aligned}
z &= \mathbf{c}_B \mathbf{B}^{-1} \mathbf{b} \\
&= \begin{bmatrix} 0 & 20 & 60 \end{bmatrix} \begin{bmatrix} 1 & 2.0 & -8.0 \\ 0 & 2.0 & -4.0 \\ 0 & -0.5 & 1.5 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 10 & 10 \end{bmatrix} \\
&= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}
\end{aligned}
$$

Duality

We can find the optimal dual solution from the optimal primal tableau as follows:

If the primal is a max problem,

$$y_i = \begin{cases} \bar{c}_{s_i}, & \text{if constraint } i \text{ is of } \le \text{ type} \\ -\bar{c}_{e_i}, & \text{if constraint } i \text{ is of } \ge \text{ type} \\ \bar{c}_{a_i} - M, & \text{if constraint } i \text{ is of } = \text{ type (max)} \\ \bar{c}_{a_i} + M, & \text{if constraint } i \text{ is of } = \text{ type (min)} \end{cases}$$

Duality

Example:

$$\max z = 3x_1 + 2x_2 + 5x_3$$

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & + & 2x_3 & \leq & 15 \\ & & 2x_2 & - & x_3 & \geq & 5 \\ 2x_1 & + & x_2 & - & 5x_3 & = & 10 \\ x_1 & , & x_2 & , & x_3 & \geq & 0 \end{array}$$

Duality

	z	x_1	x_2	x_3	s_1	e_2	a_2	a_3	RHS
z	1	0	0	0	$\frac{51}{23}$	$\frac{58}{23}$	$M - \frac{58}{23}$	$M + \frac{9}{23}$	$\frac{565}{23}$
x_3	0	0	0	1	$\frac{4}{23}$	$\frac{5}{23}$	$-\frac{5}{23}$	$-\frac{2}{23}$	$\frac{15}{23}$
x_2	0	0	1	0	$\frac{2}{23}$	$-\frac{9}{23}$	$\frac{9}{23}$	$-\frac{1}{23}$	$\frac{65}{23}$
x_1	0	1	0	0	$\frac{9}{23}$	$\frac{17}{23}$	$-\frac{17}{23}$	$\frac{7}{23}$	$\frac{120}{23}$

Duality

Since 1st constraint is of type $\leq$, $y_1 = \bar{c}_{s_1} = \frac{51}{23}$

Since 2nd constraint is of type $\geq$, $y_2 = -\bar{c}_{e_2} = -\frac{58}{23}$

Since 3rd constraint is of type $=$, $y_3 = \bar{c}_{a_3} - M = M + \frac{9}{23} - M = \frac{9}{23}$

Duality

Shadow Prices

Definition (Shadow Price):

The shadow price of the ith constraint is the amount by which the optimal z value is improved if b_i is increased by 1 assuming that the current basis remains optimal.

Duality

Shadow Prices

We can use the Dual Theorem to find the shadow prices. Consider the Dakota Example, and let us find the shadow price of the second constraint. We have

$$\mathbf{c}_B \mathbf{B}^{-1} = [y_1 \quad y_2 \quad y_3] = [0 \quad 10 \quad 10]$$

From the dual theorem, increasing the right hand side of the second constraint by 1, from 20 to 21, the objective improves by

$$\left.\begin{aligned} b_2 = 20 \Rightarrow z_1 = 48y_1 + 20y_2 + 8y_3 \\ b_2 = 21 \Rightarrow z_2 = 48y_1 + 21y_2 + 8y_3 \end{aligned}\right\} \Rightarrow z_2 - z_1 = y_2 = 10$$

which shows the shadow price of the ith constraint in a max problem is the value of the ith dual variable.

Our proof of the Dual Theorem demonstrated the following result: Assuming that a set of basic variables B is feasible, then B is optimal if and only if the associated dual solution ($\mathbf{c}_B\mathbf{B}^{-1}$) is dual feasible.

This result can be used for an alternative way of doing the following types of sensitivity analysis:

- Changing the objective coefficient of a non-basic variable
- Changing the column of a non-basic variable
- Adding a new variable (activity)

Duality

Sensitivity Analysis

Our proof of the Dual Theorem demonstrated the following result: Assuming that a set of basic variables B is feasible, then B is optimal if and only if the associated dual solution ($\mathbf{c}_B\mathbf{B}^{-1}$) is dual feasible. This result can be used for an alternative way of doing the following types of sensitivity analysis:

- Changing the objective coefficient of a non-basic variable
- Changing the column of a non-basic variable
- Adding a new variable (activity)

Duality

Sensitivity Analysis

Consider the Dakota Example:

$$\max z = 60x_1 + 30x_2 + 20x_3$$

$$\begin{array}{rcrcrcr}
8x_1 & + & 6.0x_2 & + & x_3 & \leq & 48 \\
4x_1 & + & 2.0x_2 & + & 1.5x_3 & \leq & 20 \\
2x_1 & + & 1.5x_2 & + & 0.5x_3 & \leq & 8 \\
x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}$$

The optimal solution of the primal is

$z = 280; x_1 = 2, x_2 = 0, x_3 = 8, s_1 = 24, s_2 = 0, s_3 = 0$

Duality

The dual of the problem is

$$\min w = 48y_1 + 20y_2 + 8y_3$$

$$\begin{array}{rcrcrcr}
8y_1 & + & 4.0y_2 & + & 2y_3 & \geq & 60 \\
6y_1 & + & 2.0y_2 & + & 1.5y_3 & \geq & 30 \\
y_1 & + & 1.5y_2 & + & 0.5y_3 & \geq & 20 \\
y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}$$

The optimal solution of the dual is

$w = 280; y_1 = 0, y_2 = 10, y_3 = 10$

Duality

Assume that we want to change the objective coefficient of x_2. Note that, we have

$$6y_1 + 2y_2 + 1.5y_3 \geq c_2$$

and since we have $y_1 = 0$, $y_2 = 10$, $y_3 = 10$, we can write

$$y_1 = 0, y_2 = 10, y_3 = 10 \Rightarrow 6y_1 + 2y_2 + 1.5y_3 = 35 \geq c_2$$

Hence, we see that as long as $c_2 \leq 35$, the current solution remains optimal.

Duality

Sensitivity Analysis

Look at the other examples where we change the column of non-basic variable and add a new variable (activity).

Complementary Slackness

The Theorem of Complementary Slackness is an important result that relates the optimal primal and dual solutions. To state this theorem, we assume that we have the following primal and dual problems:

Complementary Slackness

Consider the following max problem (normal max problem):

$$\max z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & \leq & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & \leq & b_2 \\
\vdots & & \vdots & & \ddots & & \vdots & \leq & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & \leq & b_m \\
x_1 & , & x_2 & , & \cdots & , & x_n & \geq & 0
\end{array}$$

Complementary Slackness

The dual of the problem is given as follows (normal min problem):

$$\min w = b_1 y_1 + b_2 y_2 + \cdots + b_m y_m$$

$$\begin{array}{ccccccccc}
a_{11} y_1 & + & a_{21} y_2 & + & \cdots & + & a_{m1} y_m & \geq & c_1 \\
a_{12} y_1 & + & a_{22} y_2 & + & \cdots & + & a_{m2} y_m & \geq & c_2 \\
\vdots & & \vdots & & \ddots & & \vdots & \geq & \vdots \\
a_{1n} y_1 & + & a_{2n} y_2 & + & \cdots & + & a_{mn} y_m & \geq & c_n \\
y_1 & , & y_2 & , & \cdots & , & y_m & \geq & 0
\end{array}$$

Moreover, let $s_1, s_2, \ldots, s_m$ and $e_1, e_2, \ldots, e_n$ be the slack and excess variables for the primal and the dual.

Complementary Slackness

We can state the Theorem of Complementary Slackness as follows:

Theorem: Complementary Slackness

If $\mathbf{x}^T = [x_1 \quad \cdots \quad x_n]^T$ and $\mathbf{y} = [y_1 \quad \cdots \quad y_m]$ be feasible primal and dual solutions, respectively, then, x is primal optimal and y is dual optimal if and only if

$$s_i y_i = 0, \quad i = 1, \ldots, m$$
$$e_j x_j = 0, \quad j = 1, \ldots, n$$

Complementary Slackness

The Theorem of Complementary Slackness implies that if a constraint in either primal or dual is non-binding ($s_i > 0$ or $e_j > 0$), then, the corresponding variable value in the corresponding problem (primal or dual) equals zero.

Complementary Slackness

Consider the Dakota Example and its dual:

$$\max z = 60x_1 + 30x_2 + 20x_3$$

$$\begin{array}{rcrcrcr}
8x_1 & + & 6.0x_2 & + & x_3 & \leq & 48 \\
4x_1 & + & 2.0x_2 & + & 1.5x_3 & \leq & 20 \\
2x_1 & + & 1.5x_2 & + & 0.5x_3 & \leq & 8 \\
x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}$$

Complementary Slackness

The dual of the problem is

$$\min w = 48y_1 + 20y_2 + 8y_3$$

$$\begin{array}{rcrcrcr}
8y_1 & + & 4.0y_2 & + & 2y_3 & \geq & 60 \\
6y_1 & + & 2.0y_2 & + & 1.5y_3 & \geq & 30 \\
y_1 & + & 1.5y_2 & + & 0.5y_3 & \geq & 20 \\
y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}$$

Complementary Slackness

The optimal solution of the dual is

$w = 280; y_1 = 0, y_2 = 10, y_3 = 10, e_1 = 0, e_2 = 5, e_3 = 0$

The optimal solution of the primal is

$z = 280; x_1 = 2, x_2 = 0, x_3 = 8, s_1 = 24, s_2 = 0, s_3 = 0$

Now, we note that

$$s_1 y_1 = s_2 y_2 = s_3 y_3 = 0$$

$$e_1 x_1 = e_2 x_2 = e_3 x_3 = 0$$

Complementary Slackness

Can we use the Theorem of Complementary Slackness to solve LPs?

Think about it!

The Dual Simplex Method

Step 1: If the RHS of each constraint is non-negative, then, stop. The optimal solution is found. Otherwise go to Step 2.

The Dual Simplex Method

Step 2: Choose the most negative basic variable as the leaving variable the row of which is the pivot row. To determine the entering variable, compute the following ratio for each variable x_j with a negative coefficient in the pivot row and choose the smallest absolute ratio, and then, perform the row operations to obtain the new solution.

$$\left|\frac{\bar{c}_{x_j}}{\bar{a}_{x_j}}\right|, \quad j: \bar{a}_{x_j} < 0$$

The Dual Simplex Method

Step 3: If there is any negative RHS corresponding to a constraint in which all coefficients are non-negative, then, the LP has no feasible solution. Otherwise return to Step 1.

The Dual Simplex Method

We can typically use the dual simplex method in the following cases:

- Finding the new optimal solution after a constraint is added.
- Finding the new optimal solution after a changing a RHS term.
- Solving normal min problem.

The Dual Simplex Method

When we add a new constraint, we can have the following cases:

- The current optimal solution satisfies the new constraint
- The current optimal solution does not satisfy the new constraint, but the LP still has a feasible solution.
- The current optimal solution does not satisfy the new constraint.

The Dual Simplex Method

The optimal solution for Dakota:

$$
\begin{array}{rrrrrrrrrrrr}
z & + & 5.00x_2 & & & & & + & 10.0s_2 & + & 10.0s_3 & = & 280 \\
 & - & 2.00x_2 & & & + & s_1 & + & 2.0s_2 & - & 8.0s_3 & = & 24 \\
 & - & 2.00x_2 & + & x_3 & & & + & 2.0s_2 & - & 4.0s_3 & = & 8 \\
x_1 & + & 1.25x_2 & & & & & - & 0.5s_2 & + & 1.5s_3 & = & 2
\end{array}
$$

If we add the constraint $x_1 + x_2 + x_3 \leq 11$, we see that the current optimal solution $z = 280; x_1 = 2, x_2 = 0, x_3 = 8$ satisfies this constraint.

The Dual Simplex Method

As an example of the second case, consider that we add the constraint $x_2 \geq 1$. We can proceed as follows by the dual simplex algorithm:

$$
\begin{array}{lrrrrrrrrrrrrrrrrrr}
z & & + & 5.00x_2 & & & & & + & 10.0s_2 & + & 10.0s_3 & & & = & 280 \\
 & & - & 2.00x_2 & & & + & s_1 & + & 2.0s_2 & - & 8.0s_3 & & & = & 24 \\
 & & - & 2.00x_2 & + & x_3 & & & + & 2.0s_2 & - & 4.0s_3 & & & = & 8 \\
 & x_1 & + & 1.25x_2 & & & & & - & 0.5s_2 & + & 1.5s_3 & & & = & 2 \\
 & & & \boxed{x_2} & & & & & & & & & + & e_4 & = & -1
\end{array}
$$

The Dual Simplex Method

$$
\begin{array}{lllllllllllllllll}
z & & & & & + & 10s_2 & + & 10s_3 & + & 5e_4 & = & 275 \\
 & & & & s_1 & + & 2s_2 & - & 8s_3 & - & 2e_4 & = & 26 \\
 & & & x_3 & & + & 2s_2 & - & 4s_3 & - & 2e_4 & = & 10 \\
 & x_1 & & & & - & \frac{1}{2}s_2 & + & \frac{3}{4}s_3 & + & \frac{5}{4}e_4 & = & \frac{3}{4} \\
 & & x_2 & & & & & & & - & e_4 & = & 1
\end{array}
$$

The Dual Simplex Method

As an example of Case 3, assume that we add a new constraint $x_1 + x_2 \geq 12$. We can then write

$$
\begin{array}{rrrrrrrrrrrrrrrr}
z & & & + & 5.00x_2 & & & & & + & 10.0s_2 & + & 10.0s_3 & & & = & 280 \\
 & & & - & 2.00x_2 & & & + & s_1 & + & 2.0s_2 & - & 8.0s_3 & & & = & 24 \\
 & & & - & 2.00x_2 & + & x_3 & & & + & 2.0s_2 & - & 4.0s_3 & & & = & 8 \\
 & & x_1 & + & 1.25x_2 & & & & & - & 0.5s_2 & + & 1.5s_3 & & & = & 2 \\
 & - & x_1 & - & x_2 & & & & & & & & & + & e_4 & = & -12
\end{array}
$$

The Dual Simplex Method

$$
\boxed{\begin{array}{lllllllll}
z & + 5.0x_2 & & & & + 10.0s_3 & & = & 280 \\
 & - x_2 & & + s_1 & & - 8.0s_3 & & = & 24 \\
 & - x_2 & + x_3 & & & - 4.0s_3 & & = & 8 \\
x_1 & + x_2 & & & & + 1.5s_3 & & = & 2 \\
 & - 0.5x_2 & & & + s_2 & + 1.5s_3 & + e_4 & = & -10
\end{array}}
$$

$$
\boxed{\begin{array}{lllllllll}
z & + 10.00x_2 & & & & + 40s_3 & + 20e_4 & = & 80 \\
 & - 2.00x_2 & & + s_1 & & - 2s_3 & + 4e_4 & = & -16 \\
 & - 2.00x_2 & + x_3 & + & & + 2s_3 & + 4e_4 & = & -32 \\
x_1 & + 1.25x_2 & & & & & - e_4 & = & 12 \\
 & 0.25x_2 & & & + s_2 & - 3s_3 & - 2e_4 & = & 20
\end{array}}
$$

The Dual Simplex Method

$$\begin{array}{|llllllllllllll|}
z & & & & 10x_3 & & & & + & 60s_3 & + & 60e_4 & = & -240 \\
& & & - & x_3 & + & s_1 & & - & 4s_3 & & & = & 16 \\
& & x_2 & - & x_3 & + & & & - & 2s_3 & - & 4e_4 & = & 32 \\
& x_1 & & + & x_3 & & & & + & 2s_3 & + & 3e_4 & = & -20 \\
& & & & & & & + \; s_2 & - & 4s_3 & - & 4e_4 & = & 36 \\
\end{array}$$

No feasible solution!

Data Envelopment Analysis (DEA)

Often we wonder if a university, hospital, restaurant, or other business is operating efficiently. The Data Envelopment Analysis (DEA) method can be used to answer this question. To illustrate how DEA works, let’s consider three hospitals. To simplify matters, we assume that each hospital “converts” two inputs into three different outputs.

Data Envelopment Analysis (DEA)

The two inputs used by each hospital are

- Input 1 = capital (measured by the number of hospital beds)
- Input 2 = labor (measured in thousands of labor hours)

The outputs produced by each hospital are

- Output 1 = hundreds of patient-days for patients under age 14
- Output 2 = hundreds of patient-days for patients between 14 and 65
- Output 3 = hundreds of patient-days for patients over 65

Data Envelopment Analysis (DEA)

Hospital	Input 1	Input 2	Output 1	Output 2	Output 3
1	5	14	9	4	16
2	8	15	5	7	10
3	7	12	4	9	13

To determine the efficiency of a hospital, we can let t_r and w_s be the value of one unit of output r and the cost of one unit of input s. The efficiency of the hospital i is then defined as the value / cost ratio.

Data Envelopment Analysis (DEA)

For our example,

$$e_1 = \frac{9t_1 + 4t_2 + 16t_3}{5w_1 + 14w_2}$$

$$e_2 = \frac{5t_1 + 7t_2 + 10t_3}{8w_1 + 15w_2}$$

$$e_3 = \frac{4t_1 + 9t_2 + 13t_3}{7w_1 + 12w_2}$$

Data Envelopment Analysis (DEA)

We also use the following assumptions:

- No hospital can be more than 100% efficient.
- An efficiency value of 1 means the corresponding hospital is efficient
- We can scale the output values.
- Each input cost and output value must be strictly positive.

Data Envelopment Analysis (DEA)

For Hospital 1,

$$\max z_1 = 9t_1 + 4t_2 + 16t_3$$

$$\begin{array}{rrrrrrrrrrrr}
- & 9t_1 & - & 4t_2 & - & 16t_3 & + & 5w_1 & + & 14w_2 & \geq & 0 \\
- & 5t_1 & - & 7t_1 & - & 10t_1 & + & 8w_1 & + & 15w_2 & \geq & 0 \\
- & 4t_1 & - & 9t_1 & - & 13t_1 & + & 7w_1 & + & 12w_2 & \geq & 0 \\
 & & & & & & & 5w_1 & + & 14w_2 & = & 1 \\
 & t_1 & & & & & & & & & \geq & 0.001 \\
 & & & t_2 & & & & & & & \geq & 0.001 \\
 & & & & & t_3 & & & & & \geq & 0.001 \\
 & & & & & & & w_1 & & & \geq & 0.001 \\
 & & & & & & & & & w_2 & \geq & 0.001
\end{array}$$

Data Envelopment Analysis (DEA)

For Hospital 2,

$$\max z_1 = 5t_1 + 7t_2 + 10t_3$$

$$\begin{array}{rrrrrrrrrrrl}
- & 9t_1 & - & 4t_2 & - & 16t_3 & + & 5w_1 & + & 14w_2 & \geq & 0 \\
- & 5t_1 & - & 7t_1 & - & 10t_1 & + & 8w_1 & + & 15w_2 & \geq & 0 \\
- & 4t_1 & - & 9t_1 & - & 13t_1 & + & 7w_1 & + & 12w_2 & \geq & 0 \\
 & & & & & & & 8w_1 & + & 15w_2 & = & 1 \\
 & t_1 & & & & & & & & & \geq & 0.001 \\
 & & & t_2 & & & & & & & \geq & 0.001 \\
 & & & & & t_3 & & & & & \geq & 0.001 \\
 & & & & & & & w_1 & & & \geq & 0.001 \\
 & & & & & & & & & w_2 & \geq & 0.001
\end{array}$$

Data Envelopment Analysis (DEA)

For Hospital 3,

$$\max z_1 = 4t_1 + 9t_2 + 13t_3$$

$$\begin{array}{rrrrrrrrrrr}
- & 9t_1 & - & 4t_2 & - & 16t_3 & + & 5w_1 & + & 14w_2 & \geq & 0 \\
- & 5t_1 & - & 7t_1 & - & 10t_1 & + & 8w_1 & + & 15w_2 & \geq & 0 \\
- & 4t_1 & - & 9t_1 & - & 13t_1 & + & 7w_1 & + & 12w_2 & \geq & 0 \\
 & & & & & & & 7w_1 & + & 12w_2 & = & 1 \\
 & t_1 & & & & & & & & & \geq & 0.001 \\
 & & & t_2 & & & & & & & \geq & 0.001 \\
 & & & & & t_3 & & & & & \geq & 0.001 \\
 & & & & & & & w_1 & & & \geq & 0.001 \\
 & & & & & & & & & w_2 & \geq & 0.001
\end{array}$$

Data Envelopment Analysis (DEA)

By solving these 3 LPs, we find that $z_1 = 1$, $z_2 = .773$ and $z_3 = 1$ (verify it), meaning that the efficiency of the hospitals are 100%, 77.3% and 100%, respectively. Interpret the solutions!

To be continued… ☺