OR-I Lecture Notes
The Simplex Algorithm

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to accompany

Operations Research: Applications and Algorithms

Outline

- Alternative Optimal Solutions
- Unbounded Solutions
- LINDO & MS Excel Solver
- Degeneracy and the Convergence of the Simplex Algorithm
- The Big M Method
- The Two-Phase Simplex Method
- Infeasible Solutions
- URS Variables
- An Overview of the Karmarkar's Algorithm

Alternative Optimal Solutions

- For some LPs, we might have more than one optimal point. In this case, we say that the LP has alternative optimal solutions.
- In this section, we will see how we can determine alternative optimal solutions during the simplex implementations.

Alternative Optimal Solutions

Example:

Consider the Dakota example as modified as follows:

$$\max z = 60x_1 + 35x_2 + 20x_3$$

s.t.

$$\begin{array}{rcrcrcr}
8x_1 & + & 6x_2 & + & x_3 & \leq & 48 \\
4x_1 & + & 2x_2 & + & 1.5x_3 & \leq & 20 \\
2x_1 & + & 1.5x_2 & + & 0.5x_3 & \leq & 8 \\
 & & x_2 & & & \leq & 5 \\
x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}$$

Alternative Optimal Solutions

Example:

Initial Simplex Tableau

	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
z	1	– 60	– 35	– 20	0	0	0	0	0
s_1	0	8	6	1	1	0	0	0	48
s_2	0	4	2	1.5	0	1	0	0	20
s_3	0	2	1.5	0.5	0	0	1	0	8
s_4	0	0	1	0	0	0	0	1	5

Alternative Optimal Solutions

Example:

Iteration 1:

	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
z	1	0	10	-5	0	0	30	0	240
s_1	0	0	0	-1	1	0	-4	0	16
s_2	0	0	-1	0.5	0	1	-2	0	4
x_1	0	1	0.75	0.25	0	0	0.5	0	4
s_4	0	0	1	0	0	0	0	1	5

Alternative Optimal Solutions

Example:

Iteration 2: Optimal

	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
z	1	0	0	0	0	10	10	0	280
s_1	0	0	-2	0	1	2	-8	0	24
x_3	0	0	-2	1	0	2	-4	0	8
x_1	0	1	1.25	0	0	-0.5	1.5	0	2
s_4	0	0	1	0	0	0	0	1	5

Alternative Optimal Solutions

Example:

Iteration 3: Alternative Optimal

	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
z	1	0	0	0	0	10	10	0	280
s_1	0	1.6	0	0	1	1.2	– 5.6	0	27.2
x_3	0	1.6	0	1	0	1.2	– 1.6	0	11.2
x_2	0	0.8	1	0	0	– 0.4	1.2	0	1.6
s_4	0	– 0.8	0	0	0	0.4	– 1.2	1	3.4

Alternative Optimal Solutions

Example:

So we can write

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.6 \\ 11.2 \end{bmatrix}$$

Thus for $0 \leq \lambda \leq 1$, all points such that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} + (1 - \lambda) \begin{bmatrix} 0 \\ 1.6 \\ 11.2 \end{bmatrix}$$

will be optimal.

Unbounded Solutions

Some LPs might have unbounded solutions. Consider the following example:

Example: Breadco Bakeries bakes two kinds of bread: French and sourdough. Each loaf of French bread can be sold for 36¢, and each loaf of sourdough bread for 30¢. A loaf of French bread requires 1 yeast packet and 6 oz. of flour; sourdough requires 1 yeast packet and 5 oz. of flour. At present, Breadco has 5 yeast packets and 10 oz. of flour. Additional yeast packets can be purchased at 3¢ each, and additional flour at 4¢/oz. Formulate and solve an LP that can be used to maximize Breadco's profits (revenues – costs).

Unbounded Solutions

x_1 = # of loaves of French bread baked

x_2 = # of loaves of sourdough bread baked

x_3 = # of yeast packets purchased

x_4 = # of ounces of flour purchased

$$\max z = 36x_1 + 30x_2 - 3x_3 - 4x_4$$

s.t.

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & x_3 & & & \leq & 5 \\ 6x_1 & + & 5x_2 & & & - & x_4 & \leq & 10 \\ x_1 & , & x_2 & , & x_3 & , & x_4 & \geq & 0 \end{array}$$

Unbounded Solutions

Initial Simplex Tableau

	z	x_1	x_2	x_3	x_4	s_1	s_2	RHS
z	1	-36	-30	3	4	0	0	0
s_1	0	1	1	-1	0	1	0	5
s_2	0	6	5	0	-1	0	1	10

Unbounded Solutions

Iteration 1

	z	x_1	x_2	x_3	x_4	s_1	s_2	RHS
z	1	0	0	3	– 1	0	6	60
s_1	0	0	1/6	– 1	1/6	1	– 1/6	10/3
x_1	0	1	5/6	0	– 1/6	0	1/6	5/3

Unbounded Solutions

Iteration 2

	z	x_1	x_2	x_3	x_4	s_1	s_2	RHS
z	1	0	2	-9	0	12	4	100
x_4	0	0	1	-6	1	6	-1	20
x_1	0	1	1	-1	0	1	0	5

Unbounded Solutions

What can you say about the simplex tableau?

We also note that if we find a vector $\mathbf{d}$ satisfying $\mathbf{cd} > 0$, the LP is unbounded.

$\mathbf{p} = [5 \quad 0 \quad 0 \quad 20 \quad 0 \quad 0]^T$ is the optimal solution vector. If we start at that point and increase x_3 by 1 unit, we need to increase x_1 and x_4 by 1 and 6 units, respectively. So, $\mathbf{d} = [1 \quad 0 \quad 1 \quad 6 \quad 0 \quad 0]^T$ is the direction of unboundedness, and we note that the LP is unbounded since

$$\mathbf{cd} = [36 \quad 30 \quad -3 \quad -4 \quad 0 \quad 0][1 \quad 0 \quad 1 \quad 6 \quad 0 \quad 0]^T = 9$$

LINDO Computer Package

We can use the LINDO computer package to solve relatively small size LPs and IPs.

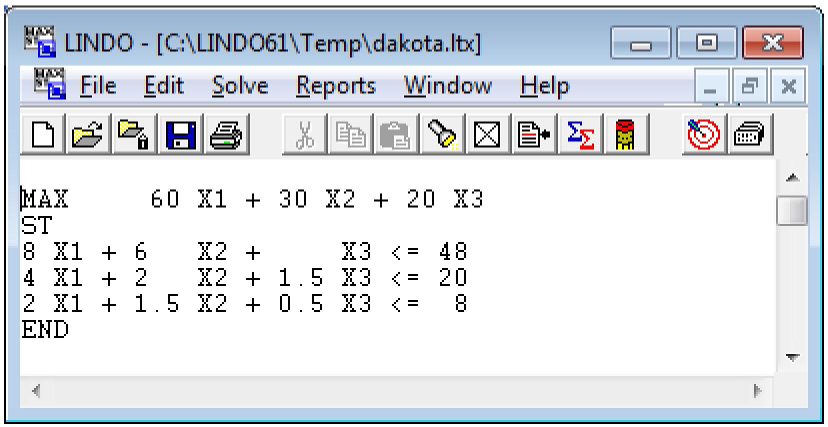
Example (Dakota Example):

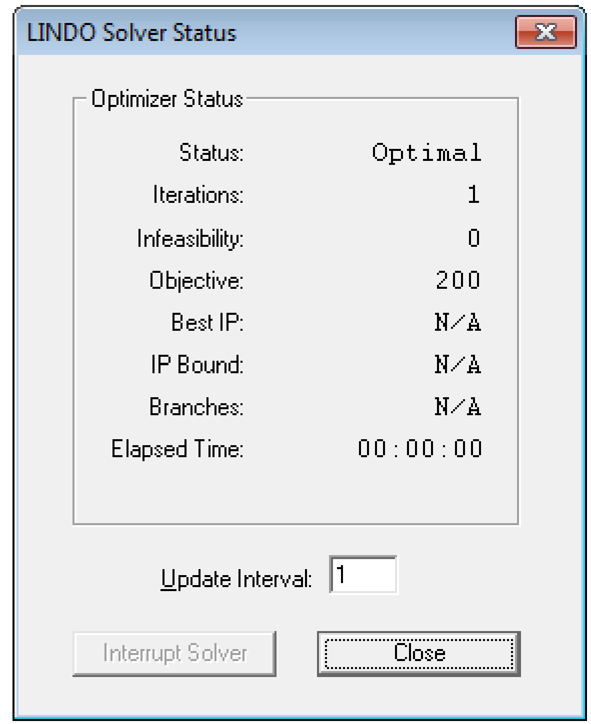
$$\max z = 60x_1 + 30x_2 + 20x_3$$

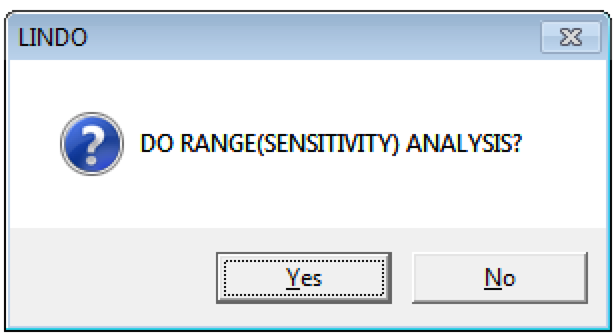
s.t.

$$\begin{array}{rcrcrcr}
8x_1 & + & 6x_2 & + & x_3 & \leq & 48 \\
4x_1 & + & 2x_2 & + & 1.5x_3 & \leq & 20 \\
2x_1 & + & 1.5x_2 & + & 0.5x_3 & \leq & 8 \\
 & & x_2 & & & \leq & 5 \\
x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}$$

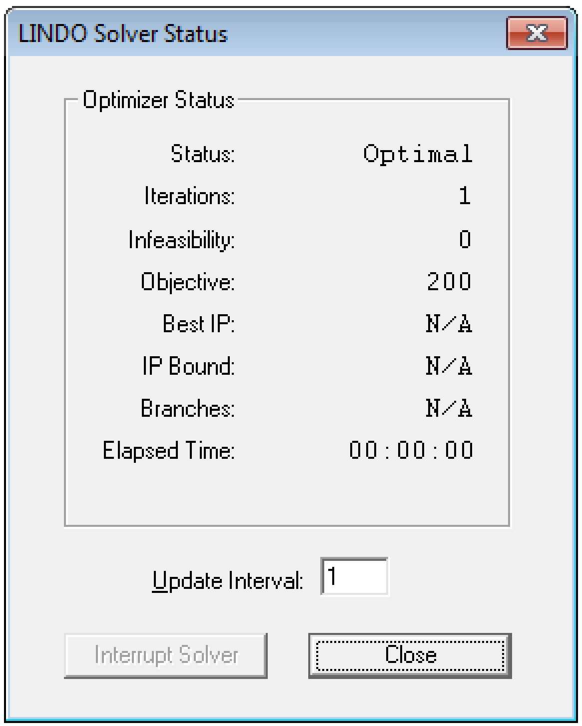
LINDO Computer Package

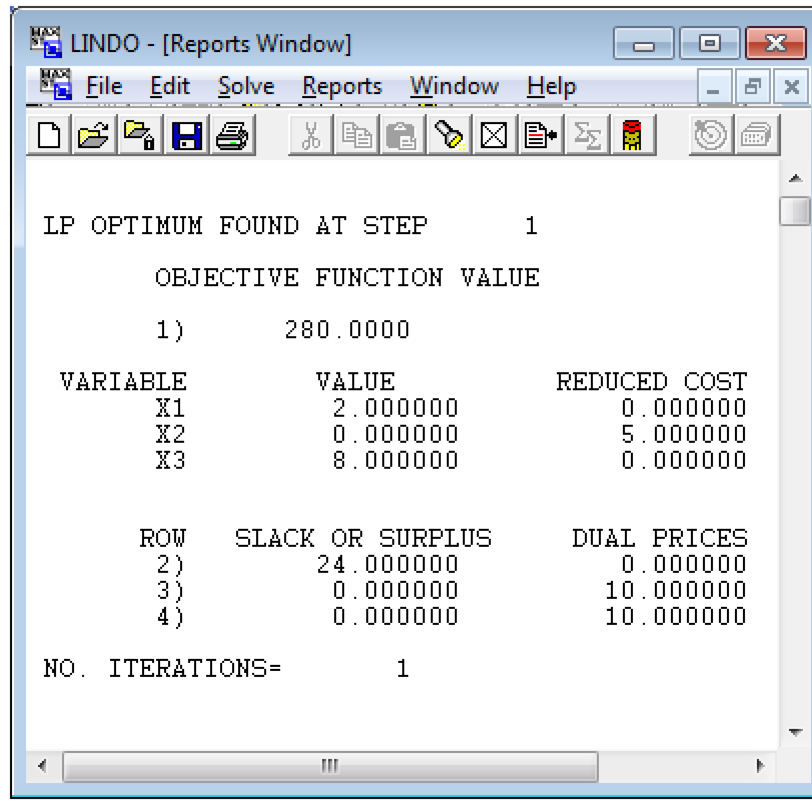




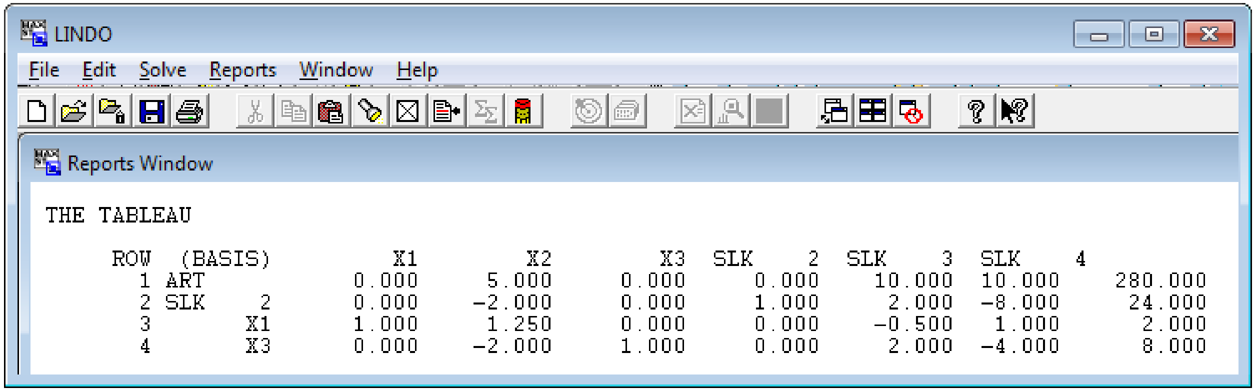


LINDO Computer Package





LINDO Computer Package



ROW	(BASIS)	X1	X2	X3	SLK 2	SLK 3	SLK 4	
1	ART	0.000	5.000	0.000	0.000	10.000	10.000	280.000
2	SLK 2	0.000	-2.000	0.000	1.000	2.000	-8.000	24.000
3	X1	1.000	1.250	0.000	0.000	-0.500	1.000	2.000
4	X3	0.000	-2.000	1.000	0.000	2.000	-4.000	8.000

Degeneracy and the Convergence of the Simplex Algorithm

- Theoretically, the simplex algorithm (as we have described it) can fail to find the optimal solution to an LP. However, LPs arising from actual applications seldom exhibit this unpleasant behavior. For the sake of completeness, however, we now discuss the type of situation in which the simplex can fail.

Degeneracy and the Convergence of the Simplex Algorithm

We can write the following expression for a max problem:

$$\begin{pmatrix} z\text{-value} \\ \text{of} \\ \text{new BFS} \end{pmatrix} = \begin{pmatrix} z\text{-value} \\ \text{of} \\ \text{current BFS} \end{pmatrix} - \begin{pmatrix} \text{entering variable value} \\ \text{of} \\ \text{new BFS} \end{pmatrix} \begin{pmatrix} \text{entering variable coefficient} \\ \text{of} \\ \text{current BFS} \end{pmatrix}$$

We can thus write the followings for a max problem:

1) $\begin{pmatrix} \text{entering variable value} \\ \text{of} \\ \text{new BFS} \end{pmatrix} > 0 \Rightarrow \begin{pmatrix} z\text{-value} \\ \text{of} \\ \text{new BFS} \end{pmatrix} > \begin{pmatrix} z\text{-value} \\ \text{of} \\ \text{current BFS} \end{pmatrix}$

2) $\begin{pmatrix} \text{entering variable value} \\ \text{of} \\ \text{new BFS} \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} z\text{-value} \\ \text{of} \\ \text{new BFS} \end{pmatrix} = \begin{pmatrix} z\text{-value} \\ \text{of} \\ \text{current BFS} \end{pmatrix}$

Degeneracy and the Convergence of the Simplex Algorithm

- An LP that satisfies property 1 is called as a non-degenerate LP.
- An LP is degenerate if it has at least one BFS in which a basic variable is equal to zero.

Degeneracy and the Convergence of the Simplex Algorithm

Example:

$$\max z = 5x_1 + 2x_2$$

s.t.

$$\begin{array}{ccccc} x_1 & + & x_2 & \leq & 6 \\ x_1 & - & x_2 & \leq & 0 \\ x_1 & , & x_2 & \geq & 0 \end{array}$$

Degeneracy and the Convergence of the Simplex Algorithm

Example:

Initial Simplex Tableau (Degenerate)

	z	x_1	x_2	s_1	s_2	RHS
z	1	-5	-2	0	0	0
s_1	0	1	1	1	0	6
s_2	0	1	-1	0	1	0

Degeneracy and the Convergence of the Simplex Algorithm

Example:

Iteration 1 (Degenerate)

	z	x_1	x_2	s_1	s_2	RHS
z	1	0	−7	0	0	0
s_1	0	0	2	1	6	6
x_1	0	1	−1	0	0	0

Degeneracy and the Convergence of the Simplex Algorithm

Example:

Iteration 2

	z	x_1	x_2	s_1	s_2	RHS
z	1	0	0	3.5	1.5	21
x_2	0	1	0	0.5	−0.5	3
x_1	0	0	1	0.5	0.5	3

Degeneracy and the Convergence of the Simplex Algorithm

- The simplex algorithm might fail to find the optimal solution to a degenerate LP.
- If an LP has many degenerate BFSs, the simplex algorithm might be very inefficient.

Degeneracy and the Convergence of the Simplex Algorithm

Example:

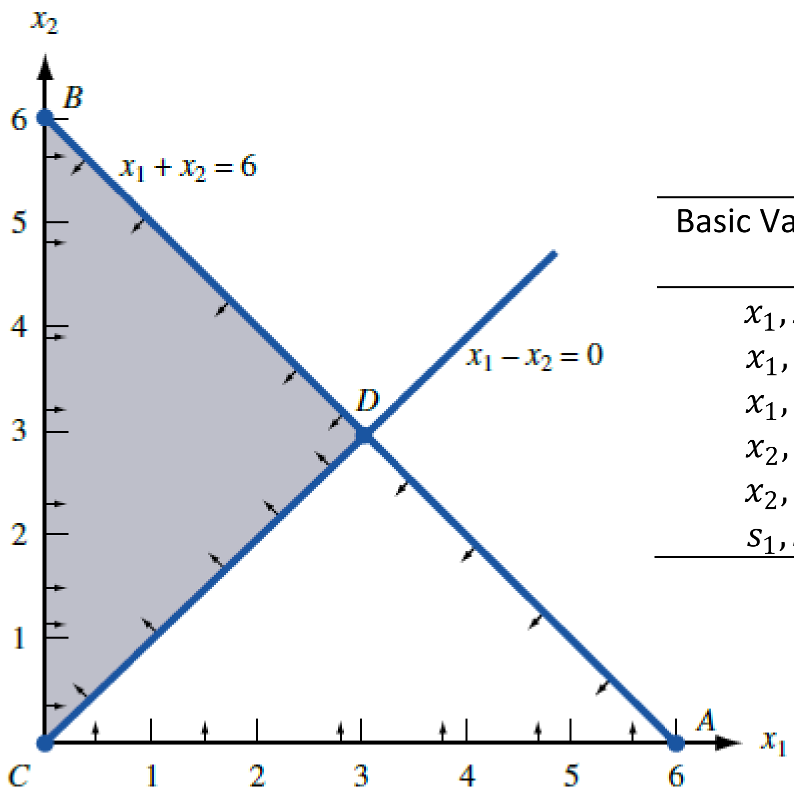
$$\max z = 5x_1 + 2x_2$$

s.t.

$$\begin{array}{ccccc} x_1 & + & x_2 & \leq & 6 \\ x_1 & - & x_2 & \leq & 0 \\ x_1 & , & x_2 & \geq & 0 \end{array}$$

Degeneracy and the Convergence of the Simplex Algorithm

Example:



Basic Variables	BFS $[x_1 \quad x_2 \quad s_1 \quad s_2]^T$	Extreme Point	Objective
x_1, x_2	[3 , 3 , 0 , 0]	D	21
x_1, s_1	[0 , 0 , 6 , 0]	C	0
x_1, s_2	[6 , 0 , 0 , – 6]	Infeasible	30
x_2, s_1	[0 , 0 , 6 , 0]	C	0
x_2, s_2	[0 , 6 , 0 , 6]	B	12
s_1, s_2	[0 , 0 , 6 , 0]	C	0

The Big M Method

Example:

- Recall that the simplex algorithm requires a starting BFS. In all the problems we have solved so far, we found a starting BFS by using the slack variables as our basic variables.
- If an LP has any greater than or equality constraints, however, a starting BFS may not be readily apparent.
- In such cases, the Big M method (or the two-phase simplex method) may be used to solve the problem.
- The Big M method, a version of the simplex algorithm that first finds a BFS by adding “artificial” variables to the problem.

The Big M Method

Example:

Example:

Bevco manufactures an orange-flavored soft drink called Oranj by combining orange soda and orange juice. Each ounce of orange soda contains 0.5 oz of sugar and 1 mg of vitamin C. Each ounce of orange juice contains 0.25 oz of sugar and 3 mg of vitamin C. It costs Bevco 2¢ to produce an ounce of orange soda and 3¢ to produce an ounce of orange juice. Bevco's marketing department has decided that each 10-oz bottle of Oranj must contain at least 20 mg of vitamin C and at most 4 oz of sugar. Use linear programming to determine how Bevco can meet the marketing department's requirements at minimum cost.

The Big M Method

Example:

Let

x_1 = # of ounces of orange soda in a bottle of Oranj
x_2 = # of ounces of orange juice in a bottle of Oranj

$$\min z = 2x_1 + 3x_2$$

s.t.

$$\begin{array}{rcrcr} \frac{1}{2}x_1 & + & \frac{1}{4}x_2 & \leq & 4 \\ x_1 & + & 3x_2 & \geq & 20 \\ x_1 & + & x_2 & = & 10 \\ x_1 & , & x_2 & \geq & 0 \end{array}$$

The Big M Method

Example:

Standard form of the LP is

$$\min z = 2x_1 + 3x_2$$

s.t.

$$\begin{array}{rcrcrcrcl}
\frac{1}{2}x_1 & + & \frac{1}{4}x_2 & + & s_1 & & & = & 4 \\
x_1 & + & 3x_2 & & & - & e_2 & = & 20 \\
x_1 & + & x_2 & & & & & = & 10 \\
x_1 & , & x_2 & & & & & \geq & 0
\end{array}$$

The Big M Method

Example:

Using the artificial variables, we can obtain an initial BFS by transforming the LP as follows:

$$\min z = 2x_1 + 3x_2 + Ma_2 + Ma_3$$

s.t.

$$\begin{array}{ccccccccccccccc}
\frac{1}{2}x_1 & + & \frac{1}{4}x_2 & + & s_1 & & & & & & & = & 4 \\
x_1 & + & 3x_2 & & & - & e_2 & + & a_2 & & & = & 20 \\
x_1 & + & x_2 & & & & & & & + & a_3 & = & 10 \\
x_1 & , & x_2 & , & s_1 & , & e_2 & , & a_2 & , & a_3 & \geq & 0
\end{array}$$

The Big M Method

Example:

Step 1-a) Modify the constraints so that the right-hand side of each constraint is nonnegative.
Step 1-b) Identify each constraint that is now “greater than or equal to” or “equal to” constraint.

Step 2) Convert each inequality to standard form using slack and excess variables.

Step 3) If there exists a constraint with “greater than or equal to” or “equal to” constraint add an artificial variable a_i to the corresponding variable.

Step 4-a) If the LP is a min problem, add $+Ma_i$ to the objective function.
Step 4-b) If the LP is a max problem, add $-Ma_i$ to the objective function.

Step 5) Because each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. This ensures that we begin with a canonical form. In choosing the entering variable, remember that M is a very large positive number.

The Big M Method

Example:

$$\min z = 2x_1 + 3x_2 + Ma_2 + Ma_3$$

s.t.

$$\begin{array}{cccccccccccccc}
\frac{1}{2}x_1 & + & \frac{1}{4}x_2 & + & s_1 & & & & & & & = & 4 \\
x_1 & + & 3x_2 & & & - & e_2 & + & a_2 & & & = & 20 \\
x_1 & + & x_2 & & & & & & & + & a_3 & = & 10 \\
x_1 & , & x_2 & , & s_1 & , & e_2 & , & a_2 & , & a_3 & \geq & 0
\end{array}$$

The Big M Method

Example:

$current\ R0$	z	$-$	$2x_1$	$-$	$3x_2$			$-$	Ma_2	$-$	Ma_3	$=$	4
$MR2$			Mx_1	$+$	$3Mx_2$	$-$	Me_2	$+$	Ma_2			$=$	$20M$
$MR3$			Mx_1	$+$	Mx_2					$+$	Ma_3	$=$	$10M$
$new\ R0$	z	$+$	$(2M-2)x_1$	$+$	$(4M-3)x_2$	$-$	Me_2					$=$	$30M$

The Big M Method

Example:

Initial Simplex Tableau

	z	x_1	x_2	s_1	e_2	a_2	a_3	RHS
z	1	$2M-2$	$4M-3$	0	$-M$	0	0	$30M$
s_1	0	1/2	1/4	1	0	0	0	4
a_2	0	1	3	0	-1	1	0	20
a_3	0	1	1	0	0	0	1	10

The Big M Method

Example:

Computation of Row 0:

		x_1	x_2	s_1	e_2	a_2	a_3	RHS
	New R2	1/3	1	0	– 1/3	1/3	0	20/3
	$(-4M+3)$ x (New R2)	$\frac{-4M+3}{3}$	$-4M+3$	0	$\frac{4M-3}{3}$	$\frac{-4M+3}{3}$	0	$\frac{-80M+60}{3}$
(1)	Current R0	$2M-2$	$4M-3$	0	$-M$	0	0	$30M$
(2)	$(-4M+3)$ x (New R2)	$\frac{-4M+3}{3}$	$-4M+3$	0	$\frac{4M-3}{3}$	$\frac{-4M+3}{3}$	0	$\frac{-80M+60}{3}$
	New R0 = (1) + (2)	$\frac{2M-3}{3}$	0	0	$\frac{M-3}{3}$	$\frac{-4M+3}{3}$	0	$\frac{10M+60}{3}$

The Big M Method

Example:

Iteration 1

	z	x_1	x_2	s_1	e_2	a_2	a_3	RHS
z	1	$\frac{2M-3}{3}$	0	0	$\frac{M-3}{3}$	$\frac{3-4M}{3}$	0	$\frac{10M+60}{3}$
s_1	0	5/12	0	1	1/12	– 1/12	0	7/3
x_2	0	1/3	1	0	– 1/3	1/3	0	20/3
a_3	0	2/3	0	0	1/3	– 1/3	1	10/3

The Big M Method

Example:

Iteration 2 (Optimal)

	z	x_1	x_2	s_1	e_2	a_2	a_3	RHS
z	1	0	0	0	$-\frac{1}{2}$	$\frac{1-2M}{2}$	$\frac{3-2M}{2}$	25
s_1	0	0	0	1	– 1/8	1/8	– 5/8	1/4
x_2	0	0	1	0	– 1/2	1/2	– 1/2	5
x_1	0	1	0	0	1/2	– 1/2	3/2	5

The Two-Phase Simplex Method

When a basic feasible solution is not readily available, the two-phase simplex method may be used as an alternative to the Big M method. In the two-phase simplex method, we add artificial variables to the same constraints as we did in the Big M method. Then we find a bfs to the original LP by solving the Phase I LP. In the Phase I LP, the objective function is to minimize the sum of all artificial variables. At the completion of Phase I, we reintroduce the original LP's objective function and determine the optimal solution to the original LP.

The Two-Phase Simplex Method

Step 1-a) Modify the constraints so that the right-hand side of each constraint is nonnegative.
Step 1-b) Identify each constraint that is now “greater than or equal to” or “equal to” constraint.

Step 2) Convert each inequality to standard form using slack and excess variables.

Step 3) If there exists a constraint with “greater than or equal to” or “equal to” constraint add an artificial variable a_i to the corresponding variable.

Step 4) By ignoring the original objective function, solve the LP with a new objective function that is just the sum of all artificial variables of the LP (refer to this as Phase I LP).

The Two-Phase Simplex Method

Step 5-a) The optimal objective value of the Phase I LP is greater than 0. In this case, the original LP has no feasible solution.

Step 5-b) The optimal objective value of the Phase I LP is 0, and no artificial variables are in the optimal basis. In this case, drop all columns for artificial variables from the optimal Phase I tableau, and solve the LP by adding the original objective function (refer to this as Phase II). The optimal solution of Phase II LP is the optimal solution of the original LP.

Step 5-c) The optimal objective value of the Phase I LP is 0, and at least one artificial variable is in the optimal basis. In this case, drop all columns for artificial variables that are non-basic and also the original variables with negative coefficients from the optimal Phase I tableau, and solve the LP by adding the original objective function (refer to this as Phase II). The optimal solution of Phase II LP is the optimal solution of the original LP.

The Two-Phase Simplex Method

$$\min z = 2x_1 + 3x_2$$

$$\begin{array}{rcrcr}
\frac{1}{2}x_1 & + & \frac{1}{4}x_2 & \leq & 4 \\
x_1 & + & 3x_2 & \geq & 20 \\
x_1 & + & x_2 & = & 10 \\
x_1 & , & x_2 & \geq & 0
\end{array}$$

The Two-Phase Simplex Method

$$\min z = 2x_1 + 3x_2$$

$$\begin{array}{rcrcrcrcl}
\frac{1}{2}x_1 & + & \frac{1}{4}x_2 & + & s_1 & & & = & 4 \\
x_1 & + & 3x_2 & & & - & e_2 & = & 20 \\
x_1 & + & x_2 & & & & & = & 10 \\
x_1 & , & x_2 & & & & & \geq & 0
\end{array}$$

The Two-Phase Simplex Method

$$\min w = a_2 + a_3$$

$$\begin{array}{ccccccccccccccc}
\frac{1}{2}x_1 & + & \frac{1}{4}x_2 & + & s_1 & & & & & & & & = & 4 \\
x_1 & + & 3x_2 & & & - & e_2 & + & a_2 & & & & = & 20 \\
x_1 & + & x_2 & & & & & & & + & a_3 & & = & 10 \\
x_1 & , & x_2 & , & s_1 & , & e_2 & , & a_2 & , & a_3 & & \geq & 0
\end{array}$$

The Two-Phase Simplex Method

current $R0$	w	$-$						$-$	a_2	$-$	a_3	$=$	0
$R2$			x_1	$+$	$3x_2$	$-$	e_2	$+$	a_2			$=$	20
$R3$			x_1	$+$	x_2					$+$	a_3	$=$	10
new $R0$	z	$+$	$2x_1$	$+$	$4x_2$	$-$	e_2					$=$	30

The Two-Phase Simplex Method

	w	x_1	x_2	s_1	e_2	a_2	a_3	RHS
w	1	2	4	0	− 1	0	0	30
s_1	0	1/2	1/4	1	0	0	0	4
a_2	0	1	3	0	− 1	1	0	20
a_3	0	1	1	0	0	0	1	10

The Two-Phase Simplex Method

	w	x_1	x_2	s_1	e_2	a_2	a_3	RHS
w	1	2/3	0	0	1/3	– 4/3	0	10/3
s_1	0	5/12	0	1	1/12	– 1/12	0	7/3
x_2	0	1/3	1	0	– 1/3	1/3	0	20/3
a_3	0	2/3	0	0	1/3	– 1/3	1	10/3

The Two-Phase Simplex Method

	w	x_1	x_2	s_1	e_2	a_2	a_3	RHS
w	1	0	0	0	0	– 1	– 1	0
s_1	0	0	0	1	– 1/8	1/8	– 5/8	1/4
x_2	0	0	1	0	– 1/2	1/2	– 1/2	5
x_1	0	1	0	0	1/2	– 1/2	3/2	5

The Two-Phase Simplex Method

There are no-artificial variables in the optimal Phase I tableau. So, drop the artificial variables and solve the problem with the original objective function, $z = 2x_1 + 3x_2 \Rightarrow z - 2x_1 - 3x_2 = 0$. Before we start, we need to apply the following operations (why?):

$$\begin{array}{|lcccccccr|}
\text{current } R0 & z & - & 2x_1 & - & 3x_2 & & & = 0 \\
3R2 & & & & & 3x_2 & - & \frac{3}{2}e_2 & = 15 \\
2R3 & & & 2x_1 & & & + & e_2 & = 10 \\
\text{new } R0 & z & & & & & - & \frac{1}{2}e_2 & = 25
\end{array}$$

The Two-Phase Simplex Method

Initial Phase II Tableau (also Optimal)

	z	x_1	x_2	s_1	e_2	RHS
z	1	0	0	0	– 1/2	25
s_1	0	0	0	1	– 1/8	1/4
x_2	0	0	1	0	– 1/2	5
x_1	0	1	0	0	1/2	5

The Two-Phase Simplex Method

$$\min z = 40x_1 + 10x_2 + 7x_5 + 14x_6$$

$$\begin{array}{rcrcrcrcrcrcl}
x_1 & - & x_2 & & & & & + & 2x_5 & & & = & 0 \\
-2x_1 & + & x_2 & & & & & - & 2x_5 & & & = & 0 \\
x_1 & & & + & x_3 & & & + & x_5 & - & x_6 & = & 3 \\
& & 2x_2 & + & x_3 & + & x_4 & + & 2x_5 & + & x_6 & = & 4 \\
x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0
\end{array}$$

$$\min w = a_1 + a_2 + a_3$$

$$\begin{array}{rcrcrcrcrcrcrcrcrcl}
x_1 & - & x_2 & & & & & + & 2x_5 & & & + & a_1 & & & & & = & 0 \\
-2x_1 & + & x_2 & & & & & - & 2x_5 & & & & & + & a_2 & & & = & 0 \\
x_1 & & & + & x_3 & & & + & x_5 & - & x_6 & & & & & + & a_3 & = & 3 \\
& & 2x_2 & + & x_3 & + & x_4 & + & 2x_5 & + & x_6 & & & & & & & = & 4 \\
x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & & & & & & & \geq & 0
\end{array}$$

The Two-Phase Simplex Method

Initial Phase I Tableau

	w	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	a_3	RHS
w	1	0	0	1	0	1	−1	0	0	0	3
a_1	0	1	−1	0	0	2	0	1	0	0	0
a_2	0	−2	1	0	0	−2	0	0	1	0	0
a_3	0	1	0	1	0	1	−1	0	0	1	3
x_4	0	0	2	1	1	2	1	0	0	0	4

The Two-Phase Simplex Method

Iteration 1 (Optimal Phase I)

	w	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	a_3	RHS
w	1	– 1	0	1	0	0	0	0	0	– 1	0
a_1	0	1	– 1	0	0	2	0	1	0	0	0
a_2	0	– 2	1	0	0	– 2	0	0	1	0	0
x_3	0	1	0	1	0	1	– 1	0	0	1	3
x_1	0	– 1	2	0	1	1	2	0	0	– 1	1

Initial Phase II Tableau

	z	x_2	x_3	x_4	x_5	x_6	a_1	a_2	RHS
w	1	– 10	0	0	– 7	– 14	0	0	0
a_1	0	– 1	0	0	2	0	1	0	0
a_2	0	1	0	0	– 2	0	0	1	0
x_3	0	0	1	0	1	– 1	0	0	3
x_1	0	2	0	1	1	2	0	0	1

The Two-Phase Simplex Method

Iteration 2 (Optimal Phase II)

	z	x_2	x_3	x_4	x_5	x_6	a_1	a_2	RHS
w	1	4	0	7	0	– 14	0	0	7
a_1	0	0	0	0	2	0	1	0	0
a_2	0	1	0	0	0	0	0	1	0
x_3	0	1	1	1/2	3/2	0	0	0	7/2
x_6	0	0	0	1/2	1/2	1	0	0	1/2

Back to Special Cases (Infeasible Solutions)

Example (Bevco-revised)

$$\min z = 2x_1 + 3x_2$$

$$\begin{array}{rcrcr} \frac{1}{2}x_1 & + & \frac{1}{4}x_2 & \leq & 4 \\ x_1 & + & 3x_2 & \geq & 36 \\ x_1 & + & x_2 & = & 10 \\ x_1 & , & x_2 & \geq & 0 \end{array}$$

Back to Special Cases (Infeasible Solutions)

Example (Bevco-revised)

$$\min z = 2x_1 + 3x_2$$

$$\begin{array}{ccccccccc}
\frac{1}{2}x_1 & + & \frac{1}{4}x_2 & + & s_1 & & & = & 4 \\
x_1 & + & 3x_2 & & & - & e_2 & = & 36 \\
x_1 & + & x_2 & & & & & = & 10 \\
x_1 & , & x_2 & & & & & \geq & 0
\end{array}$$

Back to Special Cases (Infeasible Solutions)

Example (Bevco-revised) – Big M

$$\min z = 2x_1 + 3x_2 + Ma_2 + Ma_3$$

$$\begin{array}{ccccccccccccccc}
\frac{1}{2}x_1 & + & \frac{1}{4}x_2 & + & s_1 & & & & & & & & & = & 4 \\
x_1 & + & 3x_2 & & & - & e_2 & + & a_2 & & & & & = & 36 \\
x_1 & + & x_2 & & & & & & & & + & a_3 & & = & 10 \\
x_1 & , & x_2 & , & s_1 & , & e_2 & , & a_2 & , & & a_3 & & \geq & 0
\end{array}$$

Back to Special Cases (Infeasible Solutions)

Example (Bevco-revised) – Big M

Initial Simplex Tableau

	z	x_1	x_2	s_1	e_2	a_2	a_3	RHS
z	1	$2M-2$	$4M-3$	0	$-M$	0	0	$46M$
s_1	0	1/2	1/4	1	0	0	0	4
a_2	0	1	3	0	−1	1	0	36
a_3	0	1	1	0	0	0	1	10

Back to Special Cases (Infeasible Solutions)

Example (Bevco-revised) – Big M

Iteration 1 (Infeasible)

	z	x_1	x_2	s_1	e_2	a_2	a_3	RHS
z	1	$-2M+1$	0	0	$-M$	0	$-4M+3$	$6M+30$
s_1	0	1/4	0	1	0	0	− 1/4	3/2
a_2	0	− 2	0	0	− 1	1	− 3	6
x_2	0	1	1	0	0	0	1	10

Back to Special Cases (Infeasible Solutions)

Example (Bevco-revised) – Two-Phase

$$\min w = a_2 + a_3$$

$$\begin{array}{ccccccccccccccc}
\frac{1}{2}x_1 & + & \frac{1}{4}x_2 & + & s_1 & & & & & & & & & = & 4 \\
x_1 & + & 3x_2 & & & - & e_2 & + & a_2 & & & & & = & 36 \\
x_1 & + & x_2 & & & & & & & & + & a_3 & & = & 10 \\
x_1 & , & x_2 & , & s_1 & , & e_2 & , & a_2 & , & a_3 & & & \geq & 0
\end{array}$$

Back to Special Cases (Infeasible Solutions)

Example (Bevco-revised) – Two-Phase

Initial Simplex Tableau

	w	x_1	x_2	s_1	e_2	a_2	a_3	RHS
w	1	2	4	0	– 1	0	0	46
s_1	0	1/2	1/4	1	0	0	0	4
a_2	0	1	3	0	– 1	1	0	36
a_3	0	1	1	0	0	0	1	10

Back to Special Cases (Infeasible Solutions)

Example (Bevco-revised) – Two-Phase

Iteration 1 (Infeasible)

	w	x_1	x_2	s_1	e_2	a_2	a_3	RHS
w	1	– 2	0	0	– 1	0	– 4	6
s_1	0	1/4	0	1	0	0	– 1/4	3/2
a_2	0	– 2	0	0	– 1	1	– 3	6
x_2	0	1	1	0	0	0	1	10

URS Variables

Example

A baker has 30 oz of flour and 5 packages of yeast. Baking a loaf of bread requires 5 oz. of flour and 1 package of yeast. Each loaf of bread can be sold for 30¢. The baker may purchase additional flour at 4¢/oz. or sell leftover flour at the same price. Formulate and solve an LP to help the baker maximize profits (revenues – costs).

Let

$x_1 =$ # of loaves of bread baked

$x_2 =$ # of ounces by which flour supply is increased by cash transactions

URS Variables

Example

$$\max z = 30x_1 - 4x_2$$

$$\begin{array}{rcrcr} 5x_1 & - & x_2 & \leq & 30 \\ x_1 & & & \leq & 5 \\ x_1 & & & \geq & 0 \\ & & x_2 & : & \text{urs} \end{array}$$

By letting $x_2 = x_2^+ - x_2^-$, and transforming the LP to standard form,

$$\max z = 30x_1 - 4x_2^+ + 4x_2^-$$

$$\begin{array}{rcrcrcrcrcr} 5x_1 & - & x_2^+ & + & x_2^- & + & s_1 & & & = & 30 \\ x_1 & + & & & & & & + & s_2 & = & 5 \\ x_1 & , & x_2^+ & , & x_2^- & , & s_1 & , & s_2 & \geq & 0 \end{array}$$

URS Variables

Example

Initial Tableau

	z	x_1	x_2^+	x_2^-	s_1	s_2	RHS
z	1	– 30	4	– 4	0	0	0
s_1	0	5	– 1	1	1	0	30
s_2	0	1	0	0	0	1	5

Iteration 1

	z	x_1	x_2^+	x_2^-	s_1	s_2	RHS
z	1	0	4	– 4	0	30	150
s_1	0	0	– 1	1	1	– 5	5
x_1	0	1	0	0	0	1	5

Iteration 2 (Optimal)

	z	x_1	x_2^+	x_2^-	s_1	s_2	RHS
z	1	0	0	0	4	10	170
x_2^-	0	0	– 1	1	1	– 5	5
x_1	0	1	0	0	0	1	5

URS Variables

Example

Mondo Motorcycles is determining its production schedule for the next four quarters. Demand for motorcycles will be as follows: quarter 1—40; quarter 2—70; quarter 3—50; quarter 4—20. Mondo incurs four types of costs as follows:

- It costs Mondo $400 to manufacture each motorcycle.
- At the end of each quarter, a holding cost of $100 per motorcycle is incurred.
- Increasing production from one quarter to the next incurs costs for training employees. It is estimated that a cost of $700 per motorcycle is incurred if production is increased from one quarter to the next.
- Decreasing production from one quarter to the next incurs costs for severance pay, decreasing morale, and so forth. It is estimated that a cost of $600 per motorcycle is incurred if production is decreased from one quarter to the next.

All demands must be met on time, and a quarter's production may be used to meet demand for the current quarter. During the quarter immediately preceding quarter 1, 50 Mondos were produced. Assume that at the beginning of quarter 1, no Mondos are in inventory. Formulate an LP that minimizes Mondo's total cost during the next four quarters.

URS Variables

Example

p_t = # of motorcycles produced during quarter t, $t = 1,2,3,4$

i_t = inventory at the end of quarter t, $t = 1,2,3,4$

x_t = amount by which quarter t production exceeds quarter t production, $t = 1,2,3,4$

Since x_t is unrestricted, we use the following transformation:

$$x_t = x_t^+ - x_t^-, t = 1,2,3,4$$

URS Variables

Example

$$\min z = 400p_1 + \cdots + 400p_4 + 100i_1 + \cdots + 100i_4 + 700x_1^+ + \cdots + 700x_4^+ + 600x_1^- + \cdots + 600x_4^-$$

$$\begin{aligned}
i_1 &= 0 + p_1 - 40 \\
i_2 &= i_1 + p_2 - 70 \\
i_3 &= i_2 + p_3 - 50 \\
i_4 &= i_3 + p_4 - 20 \\
x_1^+ - x_1^- &= p_1 - 50 \\
x_2^+ - x_2^- &= p_2 - p_1 \\
x_3^+ - x_3^- &= p_3 - p_2 \\
x_4^+ - x_4^- &= p_4 - p_3 \\
i_t &\geq 0, \forall t \\
p_t &\geq 0, \forall t \\
x_t^{\mp} &\geq 0, \forall t
\end{aligned}$$

Karmarkar’s Algorithm

Karmarkar’s method requires that the LP be in the following form:

$$\min z = \mathbf{cx}$$

$$\begin{aligned} \mathbf{Kx} &= 0 \\ \sum_{i=1}^{n} x_i &= 1 \\ \mathbf{x} &\geq 0 \\ \mathbf{x}^0 &= \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \end{bmatrix} \in S \\ z^* &= 0 \end{aligned}$$

Karmarkar’s Algorithm

- Karmarkar’s method uses transformed variables $y_1, y_2, \dots, y_n$ to transform the current point to the center of feasible space.
- Karmarkar’s method has been shown to be a polynomial time algorithm. This implies that if an LP of size n is solved by Karmarkar’s method, then there exist positive numbers a and b such that for any n, an LP of size n can be solved in a time of at most an^b.
- In contrast to Karmarkar’s method, the simplex algorithm is an exponential time algorithm for solving LPs. If an LP of size n is solved by the simplex, then there exists a positive number c such that for any n, the simplex algorithm will find the optimal solution in a time of at most $c2^n$.
- For large enough n, since $c2^n > an^b$, in theory a polynomial time algorithm is superior to an exponential time algoritm.
- Preliminary testing of Karmarkar’s method (by Karmarkar) has shown that for large LPs arising in actual application, this method may be up to 50 times as fast as the simplex algorithm.

End of Lecture