Operations Research I

Lecture I: Simplex Algorithm

to accompany

Operations Research: Applications and Algorithms by Wayne L. Winston

Prepared by Fatih Cavdur

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How to Convert an LP to Standard Form

- We have seen that we can solve LPs with 2 or 3 variables graphically.
- What if it has more?
- To solve larger LPs, we can use the simplex algorithm.
- We can use the simplex algorithm to solve LPs with thousands of variables and constraints.

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How to Convert an LP to Standard Form

- We have seen that an LP can have both equality and inequality constraints.
- It also can have variables that are required to be non-negative, non-positive and unrestricted in sign (urs).
- Before the simplex algorithm can be used to solve an LP, the LP must be converted into an equivalent problem in which all constraints are equations and all variables are nonnegative.
- Such an LP is said to be in standard form.

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How to Convert an LP to Standard Form (Example)

Leather Limited manufactures two types of belts: the deluxe model and the regular model. Each type requires 1 sq. yd. of leather. A regular belt requires 1 hour of skilled labor, and a deluxe belt requires 2 hours. Each week, 40 sq. yd. of leather and 60 hours of skilled labor are available. Each regular belt contributes $3 to profit and each deluxe belt, $4. If we let

$$x_1 = \text{\# of deluxe belts produced weekly}$$

$$x_2 = \text{\# of regular belts produced weekly}$$

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How to Convert an LP to Standard Form (Example)

Corresponding LP is given as

$$\max z = 4x_1 + 3x_2$$

s.t.

$$\begin{aligned} x_1 + x_2 &\leq 40 \\ 2x_1 + x_2 &\leq 60 \\ x_1 , x_2 &\geq 0 \end{aligned}$$

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How to Convert an LP to Standard Form (Example)

For each $\leq$ constraint, to convert them to equations, we add a non-negative slack variable s_i and obtain the following LP in standard form:

$$\max z = 4x_1 + 3x_2$$

s.t.

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & s_1 & & & = & 40 \\ 2x_1 & + & x_2 & + & & + & s_2 & = & 60 \\ x_1 & , & x_2 & , & s_1 & , & s_2 & \geq & 0 \end{array}$$

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How to Convert an LP to Standard Form (Another Example)

Consider the following LP:

$$\max z = 20x_1 + 15x_2$$

s.t.

$$\begin{array}{rcrcr}
x_1 & & & \leq & 100 \\
 & & x_2 & \leq & 100 \\
50x_1 & + & 35x_2 & \leq & 6{,}000 \\
20x_1 & + & 15x_2 & \geq & 2{,}000 \\
x_1 & , & x_2 & \geq & 0
\end{array}$$

How to Convert an LP to Standard Form (Another Example)

- In such cases where we have a $\geq$ constraint, we use a non-negative excess variable, e_i, to convert the inequality to an equation.

$$\max z = 20x_1 + 15x_2$$

s.t.

$$\begin{array}{rcrcrcrcrcrcr}
x_1 & & & + & s_1 & & & & & & & = & 100 \\
 & & x_2 & & & + & s_2 & & & & & = & 100 \\
50x_1 & + & 35x_2 & & & & & + & s_3 & & & = & 6{,}000 \\
20x_1 & + & 15x_2 & & & & & & & - & e_4 & = & 2{,}000 \\
x_1 & , & x_2 & , & s_1 & & s_2 & , & s_3 & , & e_4 & \geq & 0
\end{array}$$

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Preview of the Simplex Algorithm

Assume that we are given an LP with n variables and m constraints in standard form. If it is a max problem, we have

$$\max z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

s.t.

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\
\ldots & & \ldots & & \ldots & & \ldots & & \ldots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \\
 & & & & & & x_i & \geq & 0\ ,\ \forall i
\end{array}$$

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Preview of the Simplex Algorithm

If we let the following:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

We have a system of linear equations with n variables and m equations as follows:

$$\mathbf{Ax} = \mathbf{b}$$

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Preview of the Simplex Algorithm

Definition: Basic Solution

Assume that we have a system of linear equations with n variables and m equations (assume $n \geq m$) as follows:

$$\mathbf{Ax} = \mathbf{b}$$

A basic solution to the system is obtained by setting $n - m$ variables equal to 0 and solving for the remaining m variables.

This assumes that setting the $n - m$ variables equal to 0 yields unique values for the remaining m variables, i.e., the columns for the remaining m variables are linearly independent.

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Preview of the Simplex Algorithm

Definition: Basic Feasible Solution

Any basic solution to the following system in which all variables are non-negative is said to be a basic feasible solution (BFS):

$$\mathbf{Ax} = \mathbf{b}$$

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Preview of the Simplex Algorithm

Theorem:

A point in the feasible region of an LP is an extreme point if and only if it is a basic feasible solution to the LP.

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Preview of the Simplex Algorithm (Example)

$$\max z = 4x_1 + 3x_2$$

s.t.

$$\begin{array}{rcrcr} x_1 & + & x_2 & \leq & 40 \\ 2x_1 & + & x_2 & \leq & 60 \\ x_1 & , & x_2 & \geq & 0 \end{array}$$

Preview of the Simplex Algorithm (Example)

The standard form of the LP is

$$\max z = 4x_1 + 3x_2$$

s.t.

$$\begin{array}{rcccccccccl} x_1 & + & x_2 & + & s_1 & & & = & 40 \\ 2x_1 & + & x_2 & & & + & s_2 & = & 60 \\ x_1 & , & x_2 & , & s_1 & , & s_2 & \geq & 0 \end{array}$$

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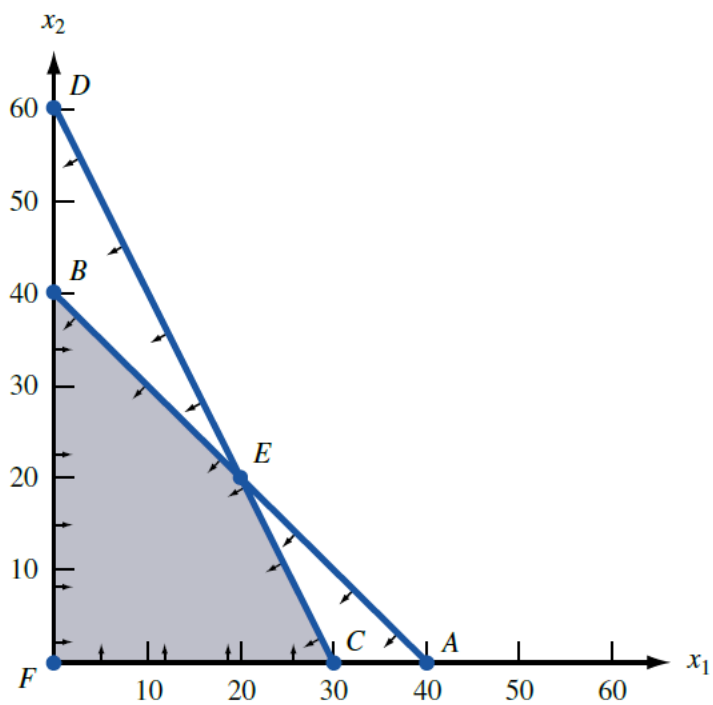
Preview of the Simplex Algorithm (Example)

The feasible region of the LP is shown below:



Extreme Points of the LP are points B, C, E and F with coordinates B (0, 40); C (30, 0); E (20, 20) and F (0, 0), respectively.

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Preview of the Simplex Algorithm (Example)

Basic Feasible Solutions (BFSs) and Extreme Points

Basic Variables	Non-Basic Variables	$[x_1, x_2, s_1, s_2]^T$				Extreme Point
x_1, x_2	s_1, s_2	20	20	0	0	E
x_1, s_1	x_2, s_2	30	0	10	0	C
x_1, s_2	x_2, s_1	40	0	0	– 20	Not a BFS
x_2, s_1	x_1, s_2	0	60	– 20	0	Not a BFS
x_2, s_2	x_1, s_1	0	40	0	20	B
s_1, s_2	x_1, x_2	0	0	40	60	F

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Preview of the Simplex Algorithm

- We can show that any BFS is an extreme point.
- Sometimes more than one set of basic variables may correspond to an extreme point. If this is true, then, we say the LP is degenerate.
- We can also show that if an LP has an optimal solution, then, it has a BFS that is optimal. To explain it, we need to define the concept of the direction of unboundedness.

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Direction of Unboundedness

Consider an LP in standard form. A non-zero vector $\mathbf{d}$ is said to be a direction of unboundedness if, for all $\mathbf{x} \in S$ and any $c \geq 0$,

$$\mathbf{x} + c\mathbf{d} \in S$$

It means, if we are in the feasible region, then, we can move as far as we want in the direction $\mathbf{d}$ and still remain in the feasible region.

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Direction of Unboundedness (Example)

$$\min z = 50x_1 + 100x_2$$

s.t.

$$\begin{array}{ccccccccccc} 7x_1 & + & 2x_2 & - & e_1 & & & = & 28 \\ 2x_1 & + & 12x_2 & & & - & e_2 & = & 24 \\ x_1 & , & x_2 & , & e_1 & , & e_2 & \geq & 0 \end{array}$$

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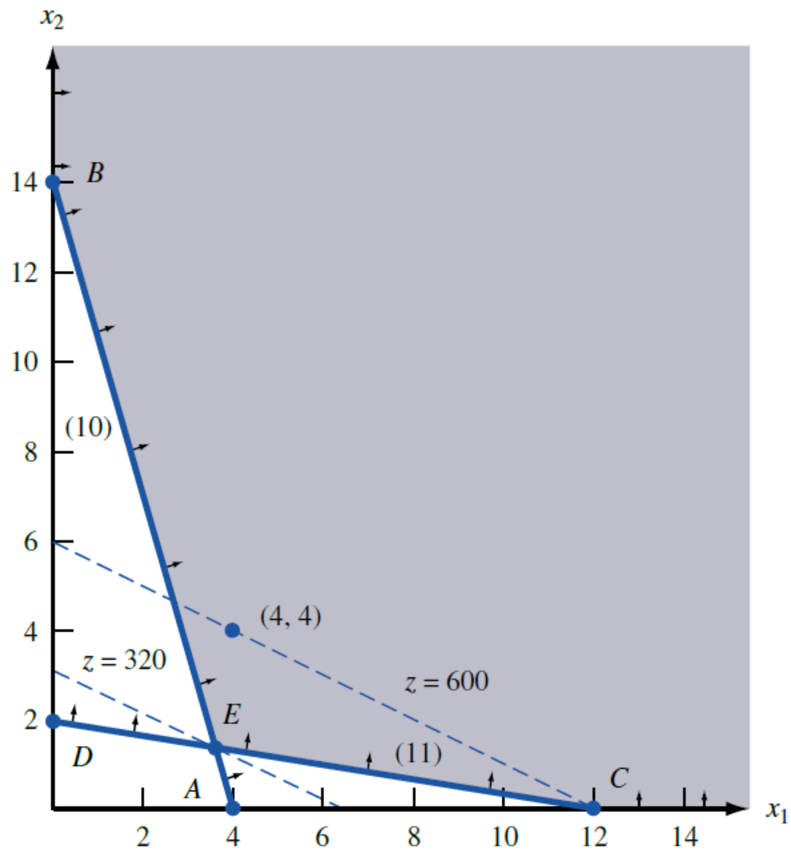
Direction of Unboundedness (Example)



If we start at any feasible point and move up and to the right at a 45-degree angle, we will remain in the feasible region. Hence, $\mathbf{d} = [1 \quad 1 \quad 9 \quad 14]$ is a direction of unboundedness for the LP.
We can show that $\mathbf{d}$ is a direction of unboundedness if and only if $\mathbf{Ad} = 0$ and $\mathbf{d} \geq \mathbf{0}$.

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Direction of Unboundedness

Theorem:

Consider an LP in standard form with BFSs $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k$. Any point $\mathbf{x}$ in the LPs feasible region may be written as follows where $\mathbf{d}$ is $\mathbf{0}$ or direction of unboundedness

$$\mathbf{x} = \mathbf{d} + \sum_{i=1}^{k} \sigma_i \mathbf{b}_i$$

and $\sum_{i=1}^{k} \sigma_i = 1$ and $\sigma_i \geq 0$.

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Direction of Unboundedness

- If the LPs feasible region is bounded, then, we have $\mathbf{d} = \mathbf{0}$, and may write that

$$\mathbf{x} = \sum_{i=1}^{k} \sigma_i \mathbf{b}_i$$

where $\sum_{i=1}^{k} \sigma_i = 1$.

- In this case, any feasible point $\mathbf{x}$ may be written as a convex combination of the LP's BFSs.

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Why does an LP have an optimal BFS?

Theorem:

If an LP has an optimal solution, then, it has an optimal BFS.

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Why does an LP have an optimal BFS?

Proof:

Let $\mathbf{x}$ be an optimal solution.
We may then write $\mathbf{x} = \mathbf{d} + \sum_{i=1}^{k} \sigma_i \mathbf{b}_i$ where $\mathbf{d}$ is $\mathbf{0}$ or direction of unboundedness and $\sum_{i=1}^{k} \sigma_i = 1$ and $\sigma_i \geq 0$.
If $\mathbf{cd} > \mathbf{0}$, then, for any $k > 0$, $k\mathbf{d} + \sum_{i=1}^{k} \sigma_i \mathbf{b}_i$ is feasible, and as k increases, z approaches to infinity which contradicts the fact that the LP has an optimal solution.
If $\mathbf{cd} < \mathbf{0}$, then, the feasible point $\sum_{i=1}^{k} \sigma_i \mathbf{b}_i$ has a larger z value than $\mathbf{x}$ which contradicts its optimality.
We have thus shown that if $\mathbf{x}$ is optimal, then, $\mathbf{cd} = \mathbf{0}$. Hence, we can write $z = \mathbf{cx} = \mathbf{cd} + \sum_{i=1}^{k} \sigma_i \mathbf{cb}_i = \sum_{i=1}^{k} \sigma_i \mathbf{cb}_i$.

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Adjacent BFSs

Definition: Adjacent BFSs

For any LP with m constraints, two BFSs are said to be adjacent BFSs if the two BFSs have $m - 1$ basic variables in common.

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General Description of the Simplex Algorithm

Step 1) Find a BFS to the LP. We refer to this as the initial BFS.

Step 2) Determine if the current BFS is an optimal solution to the LP:

(a) If it is STOP.
(b) If it is not, then, find an adjacent BFS that has a better objective function value.

Step 3) Return to Step (2) using the new BFS as the current BFS.

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General Description of the Simplex Algorithm

For an LP with n variables and m constraints, we may find at most

$$\binom{n}{m} = \frac{n!}{(n-m)!\,m!}$$

basic solutions some of which might be infeasible (not all BFSs). By enumerating these solutions to find the optimal for instance for an LP with 20 variables and 10 constraints, we need to consider $\binom{20}{10} = 184{,}756$ solutions.

Fortunately, we have the simplex algorithm which can usually find the optimal solution after examining fewer than $3m$ BFSs.

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The Simplex Algorithm

Step 1) Convert the LP to standard form.

Step 2) Obtain a BFS.

Step 3) Determine if the current BFS is optimal. If it is, then, STOP.

Step 4) If the current BFS is not optimal, determine which non-basic variable (NBV) should become a basic variable (BV) and which BV should become a NBV to find a BFS with a better objective function value.

Step 5) Find the new BFS using elementary row operations (EROs) and go back to Step (3).

The Simplex Algorithm (Example)

$$\max z = 60x_1 + 30x_2 + 20x_3$$

s.t.

$$\begin{array}{rcrcrcr}
8x_1 & + & 6x_2 & + & x_3 & \leq & 48 \\
4x_1 & + & 2x_2 & + & 1.5x_3 & \leq & 20 \\
2x_1 & + & 1.5x_2 & + & 0.5x_3 & \leq & 8 \\
 & & x_2 & & & \leq & 5 \\
x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}$$

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The Simplex Algorithm (Example)

We first convert the LP to standard form:

$$\max z = 60x_1 + 30x_2 + 20x_3 = 0 \Rightarrow z - 60x_1 - 30x_2 - 20x_3 = 0$$

We can express the LP as follows:

$$\boxed{\begin{array}{lcrcrcrcrcrcrcrcr}
z & - & 60x_1 & - & 30x_2 & - & 20x_3 & & & & & & & & & = & 0 \\
\\
 & & 8x_1 & + & 6x_2 & + & x_3 & + & s_1 & & & & & & & = & 48 \\
 & & 4x_1 & + & 2x_2 & + & 1.5x_3 & & & + & s_2 & & & & & = & 20 \\
 & & 2x_1 & + & 1.5x_2 & + & 0.5x_3 & & & & & + & s_3 & & & = & 8 \\
 & & & & x_2 & & & & & & & & & + & s_4 & = & 5
\end{array}}$$

Here, we see that $s_1 = 48$, $s_2 = 20$, $s_3 = 8$, $s_4 = 5$, and $x_i = 0$, for all i.

The above form is called as the canonical form.

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The Simplex Algorithm (Example)

Is the current BFS optimal?

Currently, we have the BVs and NBVs as follows:

$B = \{s_1, s_2, s_3, s_4\}$ and $N = \{x_1, x_2, x_3\}$. Now, consider the objective function; $\max z = 60x_1 + 30x_2 + 20x_3$.

What happens if you change x_1 from 0 to 1? Will it improve the objective function? What about x_2 and x_3? Which is better if we want to increase the value of the objective function?

In a max problem, it is the NBV with the most negative coefficient in row 0. What about in a min problem?

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The Simplex Algorithm (Example)

Entering Variable

- Since it increases z by 60 units with a unit increase, we choose to enter x_1 as the entering variable. But how large can it be?
- Note that increasing x_1 changes the values of BVs which may cause some of them being negative.
- Hence, we can increase x_1 as long as none of the other BVs become negative as follows:

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The Simplex Algorithm (Example)

From the first constraint, we can only increase x_1 so that

$$8x_1 + 6x_2 + x_3 + s_1 = 8x_1 + s_1 = 48 \Rightarrow 8x_1 \leq 48 \Rightarrow x_1 \leq 6$$

By checking for the other constrains, we obtain that

- From the 1st constraint, $x_1 \leq \frac{48}{8} = 6$
- From the 2nd constraint, $x_1 \leq \frac{20}{4} = 5$
- From the 3rd constraint, $x_1 \leq \frac{8}{2} = 4$
- From the 4th constraint, $x_1 < \infty$

Hence, $x_1 = \min\left\{\frac{48}{8}, \frac{20}{4}, \frac{8}{2}, \infty\right\} = 4$

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The Simplex Algorithm (Example)

Pivoting Row, Pivot Column and Pivot Term

$$
\begin{array}{|ccccccccccccccc|}
\hline
z & - & 60x_1 & - & 30x_2 & - & 20x_3 & & & & & & & = & 0 \\
\\
& & 8x_1 & + & 6x_2 & + & x_3 & + & s_1 & & & & & = & 48 \\
& & 4x_1 & + & 2x_2 & + & 1.5x_3 & & & + \; s_2 & & & & = & 20 \\
& & \boxed{2x_1} & + & 1.5x_2 & + & 0.5x_3 & & & & + \; s_3 & & & = & 8 \\
& & & & x_2 & & & & & & & + & s_4 & = & 5 \\
\hline
\end{array}
$$

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The Simplex Algorithm (Example)

$$
\begin{array}{rrrrrrrrrrrrrrrr}
z & - & 60x_1 & - & 30x_2 & - & 20x_3 & & & & & & & & = & 0 \\
 & & 8x_1 & + & 6x_2 & + & x_3 & + & s_1 & & & & & & = & 48 \\
 & & 4x_1 & + & 2x_2 & + & 1.5x_3 & & & + & s_2 & & & & = & 20 \\
 & & 2x_1 & + & 1.5x_2 & + & 0.5x_3 & & & & & + & s_3 & & = & 8 \\
 & & & + & x_2 & & & & & & & & & + \; s_4 & = & 5
\end{array}
$$

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The Simplex Algorithm (Example)

Initial Simplex Tableau

	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
z	1	– 60	– 30	– 20	0	0	0	0	0
s_1	0	8	6	1	1	0	0	0	48
s_2	0	4	2	1.5		1	0	0	20
s_3	0	2	1.5	0.5	0	0	1	0	8
s_4	0	0	1	0	0	0	0	1	5

x_1 enters, s_3 leaves

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The Simplex Algorithm (Example)

Current Pivot Row (Row 3):

	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
s_3	0	2	1.5	0.5	0	0	1	0	8

New Pivot Row (Row 3):

	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
x_1	0	1	0.75	0.25	0	0	0.5	0	4

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The Simplex Algorithm (Example)

Current Objective Row (Row 0):

	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
z	1	– 60	– 30	– 20	0	0	0	0	0

New Objective Row (Row 0):

	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
z	1	0	15	– 5	0	0	30	0	240

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The Simplex Algorithm (Example)

Current Row 1:

	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
s_1	0	8	6	1	1	0	0	0	48

New Row 1:

	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
s_1	0	0	0	-1	1	0	-4	0	16

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The Simplex Algorithm (Example)

Current Row 2:

	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
s_2	0	4	2	1.5		1	0	0	20

New Row 2:

	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
s_2	0	0	−1	0.5	0	1	−2	0	4

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The Simplex Algorithm (Example)

Current Row 4:

	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
s_4	0	0	1	0	0	0	0	1	5

New Row 4:

	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
s_4	0	0	1	0	0	0	0	1	5

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The Simplex Algorithm (Example)

1st Iteration

	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
z	1	0	15	– 5	0	0	30	0	240
s_1	0	0	0	– 1	1	0	– 4	0	16
s_2	0	0	– 1	0.5	0	1	– 2	0	4
x_1	0	1	0.75	0.25	0	0	0.5	0	4
s_4	0	0	1	0	0	0	0	1	5

x_3 enters, s_2 leaves

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The Simplex Algorithm (Example)

2nd Iteration

	z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS
z	1	0	5	0	0	10	10	0	280
s_1	0	0	– 2	0	1	2	– 8	0	24
x_3	0	0	– 2	1	0	2	– 4	0	8
x_1	0	1	1.25	0	0	– 0.5	1.5	0	2
s_4	0	0	1	0	0	0	0	1	5

Optimal!

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The Simplex Algorithm (for a minimization example)

$$\max z = 2x_1 - 3x_2$$

s.t.

$$\begin{array}{ccccc} x_1 & + & x_2 & \leq & 4 \\ x_1 & - & x_2 & \leq & 6 \\ x_1 & , & x_2 & \geq & 0 \end{array}$$

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The Simplex Algorithm (Example)

Initial Simplex Tableau

	z	x_1	x_2	s_1	s_2	RHS
z	1	-2	3	0	0	0
s_1	0	1	1	1	0	4
s_2	0	1	-1	0	1	6

x_2 enters, s_1 leaves

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The Simplex Algorithm (Example)

Initial Simplex Tableau

	z	x_1	x_2	s_1	s_2	RHS
z	1	-5	0	-3	0	-12
x_2	0	1	1	1	0	4
s_2	0	2	0	1	1	10

Optimal!

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Using the Simplex Algorithm to Solve Minimization Problems

Note that we can always use the following transformation and solve the transformed problem:

$$\min z = -\max(-z)$$

or

$$\max z = -\min(-z)$$

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End of Lecture

To be continued... ☺

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