Systems Simulation
Chapter 12: Comparison and Evaluation of Alternative System Designs

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Introduction

- Comparison of two system designs is easier than multiple comparisons.
- For comparison of two system designs, we can use *independent sampling* and *correlated sampling* approaches.
- For multiple comparisons, *Benferroni approach* can be used for up to 20 designs.
- For more designs, it is sometimes possible to use a *meta-model*.
- Sometimes we use simulation for optimization (optimization via simulation) to find the best system parameters.

Comparison of Two-System Designs

- In a queuing system, two possible queuing disciplines.
- In a supply-chain, two possible ordering policies.
- In a job-shop, two possible scheduling rules.
- Many other examples...

Comparison of Two-System Designs

- The method of replication will be used to analyze the output data.
- The man performance measure for system i will be denoted by θ_i, $i = 1, 2$.
- If it is a steady-state simulation, we assume that appropriate techniques (such as data deletion etc.) have already been used to ensure that the point estimators are unbiased estimators of the performance measures.
- We want to obtain point and interval estimates of the difference in mean performance, namely $\theta_1 - \theta_2$.

Example

A vehicle-safety inspection station performs three jobs: (1) brake check, (2) headlight check and (3) steering check. The present system has three stalls in parallel. When a vehicle enters a stall, an attendant makes all three inspections. From past data, we know that the arrival process is Poisson with rate 9.5 per hour and the three checks are normal with means of 6.5, 6 and 5.5 minutes, respectively, and with a common standard deviation of 0.5 minutes. No queue limit for waiting vehicles!

Example

An alternative system design suggests that each vehicle will pass through three work stations in series. No space between the vehicles in the station! Therefore, a vehicle must exit before the next vehicle can move ahead. Now we have more specialized inspectors, and thus, mean inspection times for each check now decreases by 10% to 5.85, 5.4 and 4.95 minutes, respectively. We want to compare the current system design with the alternative.

Example

System	R_1	R_2	...	R_i	Sample Mean	Sample Variance
1	Y_{11}	Y_{21}	...	$Y_{R_1 1}$	$\bar{Y}_{.1}$	S_1^2
2	Y_{12}	Y_{22}	...	$Y_{R_2 2}$	$\bar{Y}_{.2}$	S_2^2

Table: Output Data

Assuming that Y_{ri} are at least approximately unbiased, we have

$$\theta_1 = E(Y_{r1}), \forall r = 1, \ \ldots R_1$$
$$\theta_2 = E(Y_{r2}), \forall r = 1, \ \ldots R_2$$

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└Comparison of Two-System Designs

Example

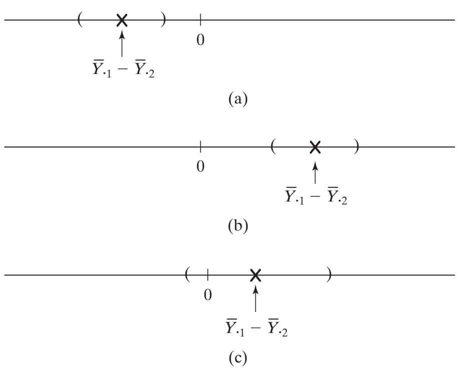

Figure: Current and Alternative System Designs

Example

By computing the confidence interval for $\theta_1 - \theta_2$, we can answer the following questions:

1. How large is the mean difference?
2. How precise is the estimator of mean difference?
3. Is there a significant difference between the two systems?
   (a) If the CI is to the left of zero, then, $\theta_1 - \theta_2 < 0 \Rightarrow \theta_1 < \theta_2$.
   (b) If the CI is to the right of zero, then, $\theta_1 - \theta_2 > 0 \Rightarrow \theta_1 > \theta_2$.
   (c) If the CI contains zero, then, no strong statistical evidence that any alternative is better or worse.

Independent Sampling

Using $\bar{Y}_{.i} = \sum_r Y_{ri}$ and the independence of replications, we have

$$V(\bar{Y}_{.i}) = \frac{V(Y_{ri})}{R_i} = \frac{\sigma_i^2}{R_i}, \; i = 1, 2$$

Since $\bar{Y}_{.1}$ and $\bar{Y}_{.2}$ are statistically independent,

$$V(\bar{Y}_{.1} - \bar{Y}_{.2}) = V(\bar{Y}_{.1}) + V(\bar{Y}_{.2}) = \frac{\sigma_1^2}{R_1} + \frac{\sigma_2^2}{R_2}$$

Sample variances are computed using

$$S_i^2 = \frac{1}{R_i - 1} \sum_{r=1}^{R_i} (Y_{ri} - \bar{Y}_{.i})^2, \; i = 1, 2$$

Independent Sampling with Equal Variances

If we have independent samples with equal variances, a pooled estimate of σ^2 is given by,

$$S_p^2 = \frac{(R_1 - 1)S_1^2 + (R_2 - 1)S_2^2}{R_1 + R_2 - 2}$$

with $v = R_1 + R_2 - 2$ d.o.f., we then have,

$$\bar{Y}_{.1} - \bar{Y}_{.2} \pm t_{\alpha/2,v}\ \text{s.e.}\ (\bar{Y}_{.1} - \bar{Y}_{.2}) = \bar{Y}_{.1} - \bar{Y}_{.2} \pm t_{\alpha/2,v} S_p \sqrt{\frac{1}{R_1} + \frac{1}{R_2}}$$

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Independent Sampling with Unequal Variances

If we have independent samples with unequal variances,

$$\text{s.e.}\ (\bar{Y}_{.1} - \bar{Y}_{.2}) = \sqrt{\frac{S_1^2}{R_1} + \frac{S_2^2}{R_2}}$$

with v d.o.f., which is approximated by,

$$v = \frac{\left(\frac{S_1^2}{R_1} + \frac{S_2^2}{R_2}\right)^2}{\frac{\left(\frac{S_1^2}{R_1}\right)^2}{R_1 - 1} + \frac{\left(\frac{S_2^2}{R_2}\right)^2}{R_2 - 1}}$$

Common Random Numbers (CRN)

- CRN means that, for each replication the same random numbers are used to simulate both systems.
- Therefore, $R_1 = R_2 = R$.
- Outputs are not independent anymore, but they are rather correlated.
- Using CRN, we want to induce a positive correlation between the outputs, and thus, achieve a variance reduction.

Common Random Numbers (CRN)

Note that,

$$\begin{aligned} V(\bar{Y}_{.1} - \bar{Y}_{.2}) &= V(\bar{Y}_{.1}) + V(\bar{Y}_{.2}) - 2 \text{ cov } (\bar{Y}_{.1}, \bar{Y}_{.2}) \\ &= \frac{\sigma_1^2}{R} + \frac{\sigma_2^2}{R} - \frac{2\rho_{12}\sigma_a\sigma_2}{R} \end{aligned}$$

We then have

$$V_{CRN} = V_{IND} - \frac{2\rho_{12}\sigma_a\sigma_2}{R} \Rightarrow V_{CRN} < V_{IND}$$

with the assumption that the CRN works properly.

Common Random Numbers (CRN)

If we let, $D_r = Y_{r1} - Y_{r2}$, which are IID by the definition of CRN,

$$\bar{D} = \frac{1}{R}\sum_{r=1}^{R} D_r \Rightarrow S_D^2 = \frac{1}{R-1}\sum_{r=1}^{R}(D_r - \bar{D})^2$$

$$= \frac{1}{R-1}\left(\sum_{r=1}^{R} D_r^2 - R\bar{D}^2\right)$$

with d.o.f. $v = R - 1$,

$$\text{s.e. } (D) = \text{ s.e. } (\bar{Y}_{.1} - \bar{Y}_{.2}) = \frac{S_D}{\sqrt{R}}$$

Example

In the vehicle-safety inspection example, we want to compare the results both using Independent Sampling (IS) and Common Random Numbers (CRN). Each vehicle to be inspected has 4 input random variables in Model 1 (Current Model).

- A_n: time between arrivals for for vehicle n and $n+1$
- $S_n^{(1)}$: brake inspection time for vehicle n
- $S_n^{(2)}$: headlight inspection time for vehicle n
- $S_n^{(3)}$: steering inspection time for vehicle n

Example

In Model 2 (Proposed Model), mean service times are decreased by 10%.

- If we use IS, time between arrivals and service time are generated using different random numbers in Model 1 and Model 2.
- If we use CRN, time between arrivals are generated the same random numbers in Model 1 and Model 2.
- For service times, although we don't want the same service times since they are 10% decreased in Model 2, we want them strongly correlated. What can we do about it?

Example

One way is to generate the service times for Model 1 first. Let them be $S_n^{(i)}$, for $i = 1, 2, 3; n = 1, 2, \ldots$. We can then use the $S_n^{(i)} - 0.1E(S_n^i)$.
Secondly, again generate the service times for Model 1 first. Let them be $S_n^{(i)}$, for $i = 1, 2, 3; n = 1, 2, \ldots$. We can then use the $E(S_n^i) + \sigma Z_n^i$.

Example

As a result, we run the simulation models for $T_E = 16$ hours with 10 replications, and average response times are recorded as a measure of performance. Results are summarized in the next slide, where M1 represents Model 1, M2-IS represents Model 2 with IS, M2-CRN represents Model 2 with CRN, and finally M2-CRN-S represents CRN with synchronization.

Example

R	M1	M2(IS)	M2(CRN)	M2(CRN-S)	M1-M2(CRN)	M1-M2(CRN-S)
1	29.59	51.62	56.47	29.55	- 26.88	0.04
2	23.49	51.91	33.34	24.26	- 9.85	- 0.77
3	25.68	45.27	35.82	26.03	- 10.14	- 0.35
4	41.09	30.85	34.29	42.64	6.80	- 1.55
5	33.84	56.15	39.07	32.45	- 5.23	1.39
10	44.00	28.44	22.44	41.49	21.56	2.51
$\bar{Y}$	37.63	43.04		-1.85	0.37	
s^2	118.90	244.33		208.94	1.74	
s.e.	6.03	6.03		4.57	0.42	

Table: Model Outputs

Example

For the independent runs M1 and M2(IS), we assume the variances are not equal. We have, $s_1^2 = 118.90$ and $s_2^2 = 244.3$, and then,

$$
\begin{aligned}
\text{s.e. } (\bar{Y}_{.1} - \bar{Y}_{.2}) &= \sqrt{\frac{S_1^2}{R_1} + \frac{S_2^2}{R_2}} \\
&= \sqrt{\frac{118.9}{10} + \frac{244.3}{10}} \\
&= 6.03
\end{aligned}
$$

then, a 95% CI will be $-18.1 \leq \theta_1 - \theta_2 \leq 7.3$.

Comparison of Several System Designs

When comparing several designs, some possible goals are as follows:

- Estimation of each parameter (performance measure), θ_i.
- Comparison of each parameter (performance measure), θ_i, to a control, θ_0, which might represent the existing system performance.
- All pairwise comparisons, $\theta_i - \theta_j$, $\forall i \neq j$.
- Selection of the best θ_i.

Benferrroni Approach

Suppose that C CIs are computed and that the ith CI with $1 - \alpha_i$. Let S_i be the statement that the ith CI contains the parameter being estimated. The Benferroni inequality then states that

$$P(\text{all statements true}) \geq 1 - \sum_{j=1}^{C} \alpha_j = 1 - \alpha_E$$

where $\alpha_E = \sum_{j=1}^{C} \alpha_j$ is the overall error probability. It means that

$$P(\text{one ore more statements false}) \leq \alpha_E$$

Two-Stage Benforroni Procedure

- Specify the practically significant difference ϵ, the probability of correct selection $1 - \alpha$, and the first-stage sample size $R_0 \geq 10$. Let $t = t_{\alpha/(K-1),R_0-1}$
- Make R_0 replications of system i to obtain $Y_{1i}, Y_{2i}, \ldots, Y_{R_0,i}$, for systems $i = 1, 2, \ldots K$.
- Calculate the first-stage sample means $\bar{Y}_{.i}$, $i = 1, 2, \ldots, K$. For all $i \neq j$, calculate the sample variance using the following expression:

$$S_{ij}^2 = \frac{1}{R_0 - 1} \sum_{r=1}^{R_0} ((Y_{ri} - Y_{rj}) - (\bar{Y}_{.i} - \bar{Y}_{.j}))^2$$

- Let $\hat{S}^2 = \max_{i \neq j} S_{ij}^2$, the largest sample variance.

Two-Stage Benforroni Procedure

- Calculate the second-stage sample size as

$$R = \max\left\{ R_0, \left\lceil \frac{t^2 \hat{S}^2}{\epsilon^2} \right\rceil \right\}$$

- Make $R - R_0$ additional replications of system i to obtain the output data $Y_{R_0+1,i}, Y_{R_0+2,i}, \ldots, Y_{R,i}$, for $i = 1, 2, \ldots, K$.
- Calculate the overall sample mean, for $i = 1, 2, \ldots, K$, as

$$\bar{\bar{Y}}_i = \frac{1}{R} \sum_{r=1}^{R} Y_{ri}$$

- Select the system with largest $\bar{\bar{Y}}_i$ as the best.

Selecting the Best

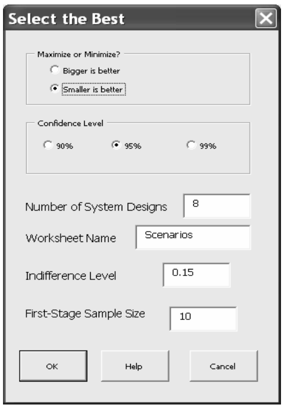

Figure: Optimization Package User Interface

Meta-Models

- Simple Linear Regression
- Multiple Linear Regression
- Non-Linear Models

Simple Linear Regression Model

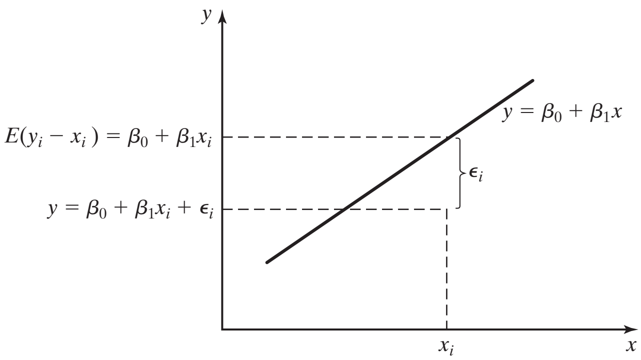

Figure: Simple Linear Regression Model

Simple Linear Regression Model

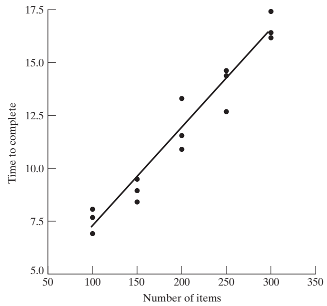

Figure: Simple Linear Regression Model

Simple Linear Regression

We have a simple linear regression model as

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \forall i = 1, 2, \ldots, n$$

By organizing the expression,

$$\epsilon_i = y_i - \beta_0 - \beta_1 x_i, \forall i = 1, 2, \ldots, n$$

The overall squared error (Sum of Squared Error - SSE) then becomes

$$\sum_{i}^{n} \epsilon_i^2 = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2, \forall i = 1, 2, \ldots, n$$

Simple Linear Regression

By finding the partial derivatives of the SSE,

$$\frac{\partial SSE}{\partial \beta_0} = -2\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)$$

$$\frac{\partial SSE}{\partial \beta_1} = -2\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)x_i$$

Simple Linear Regression

To find regression parameters,

$$\frac{\partial SSE}{\partial \beta_0} = 0 \Rightarrow n\beta_0 + \beta_1\sum_{i=1}^{n}x_i = \sum_{i=1}^{n}y_i$$

$$\frac{\partial SSE}{\partial \beta_1} = 0 \Rightarrow \beta_0\sum_{i=1}^{n}x_i + \beta_1\sum_{i=1}^{n}x_i^2 = \sum_{i=1}^{n}x_i y_i$$

Simple Linear Regression

By solving these,

$$\beta_1 = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i^2 \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{(x_i - \bar{x})^2}$$

$$\beta_0 = \frac{\sum_{i=1}^{n} y_i - \beta_1 \sum_{i=1}^{n} x_i}{n} = \bar{y} - \beta_1 \bar{x}$$

Other Models

Some other regression models can be considered.

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \epsilon$$

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 + \epsilon$$

Optimization

For m controllable design (decision) variables, we want to minimize or maximize some performance measure(s) (objective function(s)) as

$$\min E(F(x_1, x_2, \ldots, x_n))$$

or

$$\max E(F(x_1, x_2, \ldots, x_n))$$

Genetic Algorithms

1. Set $j = 0$.
2. Initialize population of size p as $\mathbf{P}(0) = \{\mathbf{x}_1(0), \ldots, \mathbf{x}_p(0)\}$ (randomly).
3. Run simulation to obtain $Y(\mathbf{x}), \forall \mathbf{x}(j) \in \mathbf{P}(j)$.
4. Select a population of p solutions from those in $\mathbf{P}(j)$.
(such that better $Y(\mathbf{x})$ are more likely)
Let it be $\mathbf{P}(j+1)$.
5. Apply crossover and mutation to the solutions in $\mathbf{P}(j+1)$.
6. Set $j = j + 1$.
Go to Step 2.

Tabu Search

1. Set $j = 0$ and $T = \emptyset$.
   Compute an initial solution $\mathbf{x}^* \in \mathbf{X}$ (randomly).
2. Run simulation to obtain $\mathbf{x}' \in \text{neighbor}(\mathbf{x}^*)$ to optimize $Y(\mathbf{x})$.
   (such that $\mathbf{x}' \notin T$)
3. If $Y(\mathbf{x}')$ better than $Y(\mathbf{x}')$, then, set $\mathbf{x}^* = \mathbf{x}'$.
4. Set $j = j + 1$ and update T.
   Go to Step 2.

Random Search

1. Initialize $C(i) = 0, \forall i$.
   Compute an initial solution i^0 and set $C(i^0) = 1$.
2. Compute another solution i' with equal probabilities.
   (such that $i' \neq i^0$)
3. Run simulation for i^0 and i' to obtain $Y(i^0)$ and $Y(i')$.
4. If $Y(i')$ is better than $Y(i^0)$, then, set $i^0 = i'$.
5. Set $C(i^0) = C(i^0) + 1, \forall i$.
   If algorithm terminates, then, select x_{i^*}.
   (such that $C(i^*) = \max_i C(i)$)
   Otherwise, go to Step 2.

Summary

- Reading HW: Chapter 12.
- Chapter 12 Exercises.